Topological and metric entropy for group and semigroup actions

Luchezar Stoyanov

University of Western Australia

Dynamical Methods in Algebra, Geometry and Topology
Udine, July 2018

1. Receptive topological entropy

Let G be a locally compact semigroup with identity e.
By a regular system in G we mean an infinite sequence

$$\alpha = (N_1, N_2, \ldots, N_n, \ldots)$$

of compact subsets of G containing e such that :

$$N_m \cdot N_n \subset N_{m+n}$$

for all natural numbers m and n.

Examples. (a) Let U be an arbitrary compact neighbourhood of e in G. Then

$$\alpha_U = (U, U^2, \ldots, U^n, \ldots)$$

is a regular system in G, called a standard system.

(b) Let $G = \mathbb{R}^m \times \mathbb{Z}^k \times K$, where m and k are non-negative integers and K is a compact semigroup. Set

$$N_n = [-n, n]^m \times [-n, n]^k \times K,$$

for all $n \geq 1$. Then

$$\alpha_0 = (N_1, \ldots, N_n, \ldots)$$

is a regular system in G.
Another example:

$$\beta_0 = (N_1^+, \ldots, N_n^+, \ldots),$$

where

$$N_n^+ = [0, n]^m \times [0, n]^k \times K.$$

This is also a regular system in the semigroup

$$S = \mathbb{R}_+^m \times \mathbb{Z}_+^k \times K.$$

Uniform actions

Let G be a topological semigroup and let (X, ρ) be a metric space.
A continuous (left) semigroup action

$$T : G \times X \longrightarrow X \quad , \quad T(g, x) = g\,x,$$

is called uniform on compact subsets of G (or just uniform), if for every $\epsilon > 0$ and every compact subset N of G there exists $\delta > 0$ such that if $x, y \in X$ and $\rho(x, y) < \delta$, then $\rho(gx, gy) < \epsilon$ for any $g \in N$.

In what follows T always denotes a uniform semigroup action.

Spanning and separated sets

Lemma 1.

Let $\epsilon > 0$ and let N be a compact subset of G.

(a) *Then for every $x \in X$ the set*

$$D_N(x, \epsilon) = D_N(x, \epsilon, T) = \{y \in X : \ \rho(gx, gy) < \epsilon \ \forall \ g \in N\}$$

is an open neighbourhood of x in X.

(b) *Let K be a compact subset of X. There exists a finite $F \subset X$ such that*

$$K \subset \bigcup_{x \in F} D_N(x, \epsilon).$$

Let

$$T : G \times X \longrightarrow X \quad , \quad T(g, x) = g\,x,$$

be a fixed uniformly continuous action.
K – a compact subset of X, $\epsilon > 0$, $N \subset G$.

As in Bowen (1971), a subset F of K is called (N, ϵ)-spanning for K if

$$K \subset \bigcup_{x \in F} D_N(x, \epsilon).$$

A subset E of X is called (N, ϵ)-separated if for any $x \neq y$ in E there exists $g \in N$ with $\rho(gx, gy) \geq \epsilon$.

Fact (easy to see): every (N, ϵ)-separated subset of K is contained in a maximal (N, ϵ)-separated subset of K which is also (N, ϵ)-spanning for K.

Let $\alpha = (N_1, N_2, \ldots, N_n, \ldots)$ be a fixed regular system in G.

For compact $K \subset X$, $\epsilon > 0$ and an integer $n \geq 1$, set :

$$r_{N_n}(\epsilon, T, K) = \min \{|F| : F \subset K, F \text{ is } (N_n, \epsilon) - \text{spanning for } K\},$$

$$s_{N_n}(\epsilon, T, K) = \max \{|E| : E \subset K, E \text{ is } (N_n, \epsilon) - \text{separated}\}.$$

Here $|A|$ is the cardinality of the set A.

For brevity

$$r_{N_n}(\epsilon, K) = r_{N_n}(\epsilon, T, K) \quad , \quad s_{N_n}(\epsilon, K) = s_{N_n}(\epsilon, T, K).$$

Easy to see:

$$r_{N_n}(\epsilon, K) \leq s_{N_n}(\epsilon, K) \leq r_{N_n}(\epsilon/2, K).$$

Then define

$$\tilde{r}(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log r_{N_n}(\epsilon, K),$$

$$\tilde{s}(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{n} \log s_{N_n}(\epsilon, K).$$

For every compact $K \subset X$ the monotone limits

$$\tilde{h}(T, K) = \tilde{h}(T, K, \alpha) = \lim_{\epsilon \searrow 0} \tilde{r}(\epsilon, K) = \lim_{\epsilon \searrow 0} \tilde{s}(\epsilon, K) \in [0, \infty]$$

exist and are equal. This will be called the **receptive topological entropy of T with respect to K** (and $\alpha = \{N_n\}$).

Definition. The **receptive topological entropy** of T (with respect to α) is defined by

$$\tilde{h}(T) = \sup\{\tilde{h}(T, K) : K \subset X \ , K \ - \text{compact}\ \}.$$

Clearly, $\tilde{h}(T) = \tilde{h}(T, \alpha)$ depends on the regular system α.

(We call it "receptive" to distinguish it from the classical topological entropy.)

Topological entropy of this kind has been considered by various people:

- E. Ghys, R. Langevin and P.Walczak (1988)
- K.H.Hofmann and L. Stoyanov (1995)
- A. Biś (2004)
- (maybe others ???)

The case $G = \mathbb{Z}$ (or $G = \mathbb{Z}_+$)

When $G = \mathbb{Z}_+$, the action of T is given by a continuous map $f : X \longrightarrow X$, that is we have $T(n, x) = f^n(x)$. Then $\tilde{h}(T) = h(f)$, the classical topological entropy of the map f, as defined by Bowen (1971) and Dinaburg (1970).

The regular system here (which appears implicitly in the definition) is given by $N_n = [0, n) \cap \mathbb{Z}$.

Kushnirenko (1967) appears to be the first who pointed out that changing the system $\{N_n\}$ produces a different entropy.
In the case of measure-preserving transformations T he introduced the so called A-entropy, where A is a particular sequence of positive integers. E.g. when $A = \{2^n\}$ he showed that the A-entropy does not coincide with the classical one.

Entropy for smooth actions on manifolds

From Hofmann-St. (1995):

Proposition 1.

Let X be a Riemannian manifold and let $T : G \times X \longrightarrow X$ be a continuous action of a semigroup G such that the map $x \mapsto gx$ is smooth for each $g \in G$. Let $\alpha = \{N_n\}_{n=1}^{\infty}$ be a regular system in G with $N_n = N_1^n$ for all $n \geq 1$. Consider X with the metric generated by the Riemannian metric of X, and let $k = \dim X$. Then

$$\tilde{h}(T) \leq \max\{0, k \log a\} < \infty,$$

where $a = \sup_{x \in X} \sup_{g \in N_1} \|d_x g\|$.

This is an analogue of Bowen-Kushnirenko's Theorem in the case $G = \mathbb{Z}_+$.

Some properties of $\tilde{h}(T)$

From Hofmann-St. (1995):

Property 1. For any uniform action we have

$$\tilde{h}(T) \geq \sup_{g \in N_1} h(g),$$

where $h(g)$ is the (classical) topological entropy of the map $g : x \mapsto gx$.

Property 2. Let

$$T : G \times X \longrightarrow X \ , \ \ S : H \times Y \longrightarrow Y$$

be uniform actions on metric spaces $(X, \rho), (Y, \sigma)$, and let

$$\alpha = (N_1, N_2, \ldots) \ , \ \ \beta = (M_1, M_2, \ldots)$$

be regular systems in G and H, respectively. Let $\varphi : G \longrightarrow H$ be a semigroup homomorphism and let $f : X \longrightarrow Y$ be a continuous map with

$$f \circ T = S \circ (\varphi \times f).$$

(a) *If f is injective, $f^{-1} : f(X) \longrightarrow X$ is uniformly continuous and $\varphi(N_n) \subset M_n$ for every n, then*

$$h_\alpha(T) \leq h_\beta(S).$$

(b) (**Conjugate actions have the same entropy**):
If f is bijective and both f and f^{-1} are uniformly continuous and $\varphi(N_n) = M_n$ for every n, then

$$\tilde{h}_\alpha(T) = \tilde{h}_\beta(S).$$

Products

Property 3. *For* $i = 1, 2$, *let*

$$T_i : G_i \times X_i \longrightarrow X_i$$

be a uniform actions of the locally compact semigroups G_i, *with given regular systems* $\alpha_i = (N_n^{(i)})_{n=1}^{\infty}$, *on the metric spaces* (X_i, ρ_i). *Consider* $X = X_1 \times X_2$ *with the metric* $\rho = \max\{\rho_1, \rho_2\}$. *Let*

$$G = G_1 \times G_2 \quad , \quad \alpha = (N_n)_{n=1}^{\infty} \quad , \quad N_n = N_n^{(1)} \times N_n^{(2)}.$$

Define the action T *by*

$$T\left((g_1, g_2), (x_1, x_2)\right) = \left(T_1(g_1, x_1), T_2(g_2, x_2)\right).$$

Products

Then

$$\max\{\tilde{h}(T_1, \alpha_1), \tilde{h}(T_2, \alpha_2)\} \leq \tilde{h}(T, \alpha) \leq \tilde{h}(T_1, \alpha_1) + \tilde{h}(T_2, \alpha_2).$$

If either X_1 or X_2 is compact, then

$$\tilde{h}(T, \alpha) = \tilde{h}(T_1, \alpha_1) + \tilde{h}(T_2, \alpha_2).$$

The proof of Property 3 is almost the same as the one in the case $G = \mathbb{Z}$ (or $G = \mathbb{Z}_+$).

Entropy by open covers

We will use a simple modification of the classical definition of topological entropy using open covers by Adler, Konheim and McAndrew (1965).

Let again $T : G \times X \longrightarrow X$ be an uniformly continuous action of the (semi)group G on a metric space (X, ρ) and let $\{N_n\}_{n\geq 1}$ be a regular system in G such that **each N_n is a finite subset** of G. Assume X is **compact**.

Recall some notation:

if $\mathcal{A} = \{A_i\}_{i\in I}$ and $\mathcal{B} = \{B_j\}_{j\in J}$ are families of subsets of X, set

$$\mathcal{A} \wedge \mathcal{B} = \{A_i \cap B_j : i \in I\,,\ j \in J\}.$$

$$\mathcal{A}^{\mathbf{n}} = \bigvee_{g\in N_n} g^{-1}\mathcal{A} = \left\{ \bigcap_{g\in N_n} g^{-1}A_{i_g} : i : N_n \longrightarrow I \right\}.$$

Entropy by open covers

For any open cover α of X let $N(\alpha)$ be the number of elements of a sub-cover of α of the smallest possible cardinality.

Proposition 2. [B,D,GB,St]

(A.Biś, D. Dikranjan, A.Giordano Bruno, L.St. 2018)

$$\tilde{h}(T) = \sup_{\alpha} \limsup_{n\to\infty} \frac{1}{n} \log N(\alpha^n),$$

where α runs over the set of all finite open covers of X.

Another result in [B,D,GB,St] shows that, assuming G is finitely generated and commutative, the receptive topological entropy $\tilde{h}(T)$ coincides with a Bowen-type concept (Bowen 1973), defined similarly to Hausdorff dimension.

Amenable groups – a special case

Definition. A discrete (infinite) countable semigroup G is called **amenable** if it admits a **Følner sequence**, i.e. a sequence $\{F_n\}_{n=1}^{\infty}$ of finite subsets of G such that

$$\lim_{n\to\infty} \frac{|F_n \,\Delta\, gF_n|}{|F_n|} = 0$$

for all $g \in G$.

Here

$$A \,\Delta\, B = (A \setminus B) \cup (B \setminus A)$$

is the symmetric difference of the sets A and B.

E.g. $G = \mathbb{Z}^n$ and $G = \mathbb{Z}_+^n$ are amenable.

Topological entropy for amenable group actions

Assume that such a group G acts continuously on a **compact** metric space (X, d).
Given a Følner sequence $\{F_n\}$ in G, define the **topological entropy** of T by

$$h(T) = \sup_{\alpha} \lim_{n\to\infty} \frac{1}{|F_n|} \log N(\alpha^n),$$

where α runs over the set of all open covers of X.
(The definition does not depend on the choice of the Følner sequence.)

Bowen-type definition of the topological entropy

For general metric spaces (X, ρ) and a compact $K \subset X$ set

$$r(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log r_{F_n}(\epsilon, K),$$

$$s(\epsilon, K) = \limsup_{n\to\infty} \frac{1}{|F_n|} \log s_{F_n}(\epsilon, K),$$

$$h(T, K) = \lim_{\epsilon \searrow 0} r(\epsilon, K) = \lim_{\epsilon \searrow 0} s(\epsilon, K) \in [0, \infty].$$

Then

$$h(T) = \sup\{h(T, K) : K \subset X \ , K \ \text{compact} \ \}$$

is called the **topological entropy** of T.
For compact X the definition agrees with the one using open covers.

A few references

A large number of authors have studied various aspects of amenable group actions and more specifically **classical** topological and metric entropies for such actions, topological pressure, etc.
A (very incomplete) list of such authors includes:

- A. A. Kirillov 1967
- J. P. Conze 1972
- Y. Katznelson and B. Weiss 1972
- D. Ruelle 1973
- F. B. Greenleaf 1973
- J. C. Kieffer 1975
- S.A. Elsanousi 1977
- E. Eberlein 1977

A few references (cont.)

- M. Misiurewicz 1977
- A.M. Stepin and A.T. Tagi-Zade 1980
- J. M. Ollagnier and D. Pinchon 1982
- J. M. Ollagnier 1985
- D. Ornstein and B. Weiss 1987
- D. Lind, K. Schmidt and T. Ward 1990
- A. Bufetov 1999
- E. Lindenstaruss 2001
- B. Weiss 2003
- L. Bowen 2010
- D. Zheng and E. Chen 2016
- D. Dikranjan, A. Fornasiero and A. Giordano Bruno 2018
- D. Dikranjan and A. Giordano Bruno 2018

Comparison between the classical and the receptive topological entropies

An Example. Let $G = \mathbb{Z}_+^k$, $k > 1$. Set

$$N_n = F_n = [0, n-1]^k \cap G.$$

Then $|F_n| = n^k$.
Assume e.g. that G acts on a compact metric space X. For any open cover α of X, the definitions give

$$\tilde{h}(T) = \sup_{\alpha} \limsup_{n\to\infty} \frac{1}{n} \log N(\alpha^n) \quad ,$$

$$h(T) = \sup_{\alpha} \limsup_{n\to\infty} \frac{1}{n^k} \log N(\alpha^n),$$

where α runs over the set of all open covers of X.

Comparison between the classical and the receptive topological entropies

Clearly, if $\tilde{h}(T) < \infty$, then $h(T) = 0$ and if $h(T) > 0$, then $\tilde{h}(T) = \infty$.

The classical entropy $h(T)$ is frequently zero:

Proposition 3

Eberlein (1977): *Let G be finitely generated and commutative and let $g_1, \ldots g_k$ be generators of G. If $h(g_i) < \infty$ for some $i = 1, \ldots, k$, then $h(T) = 0$.*

Here $h(g_i)$ is the topological entropy of the map $g_i : X \longrightarrow X$.

Comparison between the classical and the receptive topological entropies

Corollary

If $T : \mathbb{Z}_+^k \times X \longrightarrow X$ is a smooth (C^1 is enough) action of $\mathbb{Z}_+^k$ on a compact Riemannian manifold X and $k > 1$, then $h(T) = 0$.

Here $\mathbb{Z}_+^k$ can be replaced by other amenable semigroups.

Classical metric entropy

Let $(X, \mathcal{M}, \mu)$ be a measure space with a probability measure μ defined on a σ-algebra $\mathcal{M}$ of subsets of X, and let

$$T : G \times X \longrightarrow X \quad , \quad T(g, x) = g\,x,$$

be a **measure-preserving action** of an amenable semigroup G on X, i.e. the map $g : X \longrightarrow X$, $x \mapsto gx$, is measure-preserving for all $g \in G$.

If $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ is a partition of X by measurable subsets, then the **entropy of $\mathcal{A}$** is defined by

$$H_\mu(\mathcal{A}) = -\sum_{i=1}^{k} \mu(A_i)\, \log \mu(A_i),$$

where by definition $0\, \log 0 = 0$.

Classical metric entropy

Let $\{F_n\}_{n=1}^{\infty}$ be e Følner sequence in G.
For every finite measurable partition $\mathcal{A}$ of X set

$$h_\mu(T,\mathcal{A}) = \limsup_{n\to\infty} \frac{1}{|F_n|} H_\mu\left(\bigvee_{g\in N_n} g^{-1}\mathcal{A}\right).$$

Then

$$h_\mu(T) = \sup_{\mathcal{A}} h_\mu(T,\mathcal{A}),$$

where the supremum is taken over all finite measurable partitions $\mathcal{A}$ of X, is called the **(metric) entropy of T**.

Classical metric entropy

This concept has been studied for a long time by various authors, and various properties have been established.

Again, it is frequently zero.

E.g. we have

Proposition 4

Let $g_1, \ldots g_k$ be generators of $G = \mathbb{Z}^k$. If $h_\mu(g_i) < \infty$ for some $i = 1, \ldots, k$, then $h_\mu(T) = 0$.

Proved by Conze (1972). Also true for any k and other (semi)groups G.

Classical Variational Principle

Assume that X is a compact space and G is a discrete finitely generated semigroup acting on X.

Let $M(X, T)$ be the **set of all T-invariant Borel probability measures** on X.

Variational Principle (VP)

$$h(T) = \sup\{h_\mu(T) : \mu \in M(X, T)\}.$$

There is a more general VP concerning topological pressure.

Classical Variational Principle - bits of history

Proofs of the VP have been given by various people in various different situations:

Ruelle (1973; for $G = \mathbb{Z}^k$ and topological pressure, under some conditions),

Elsanousi (1977; for $G = Z^2$ and topological pressure),

Misiurewicz (1977; for $G = \mathbb{Z}_+^k$ and topological pressure),

Ollagnier and Pinchon (1982; for amenable groups and topological pressure),

and others.

The case $G = \mathbb{Z}_+$ (or $G = \mathbb{Z}$) was done earlier – L. W. Goodwyn, E.I.Dinaburg, T.N.T.Goodman.

Receptive metric entropy

Let again $(X, \mathcal{M}, \mu)$ be a measure space with a probability measure μ, and let $T : G \times X \longrightarrow X$ be a measure-preserving action of a semigroup G on X.
Let $\Gamma = (N_1, N_2, \ldots, N_n, \ldots)$ be e regular system in G, **consisting of finite subsets** of G.
For every finite measurable partition $\mathcal{A}$ of X set

$$\tilde{h}_\mu(T, \mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n} H_\mu \left(\bigvee_{g \in N_n} g^{-1} \mathcal{A} \right) .$$

Then

$$\tilde{h}_\mu(T) = \sup_{\mathcal{A}} \tilde{h}_\mu(T, \mathcal{A}),$$

where the supremum is taken over all finite measurable partitions $\mathcal{A}$ of X, is called the **receptive metric entropy of T** with respect to the system Γ .

Receptive metric entropy

The special case $G = \mathbb{Z}^2$ with the standard regular system was briefly studied by L. Todorovich (2009).

The receptive metric entropy is studied in more details in [A,D,GB,St] (work in progress), where various natural properties are established:

- conjugacy invariance
- formula for product actions (same as for $G = \mathbb{Z}_+$), etc.

Variational Principle for Receptive Entropies – a few remarks

Let again $(X, \mathcal{M}, \mu)$ be a measure space with a probability measure μ, let $T : G \times X \longrightarrow X$ be a measure-preserving action of a semigroup G on X, and let $\Gamma = (N_n)$ be e regular system in G, consisting of finite subsets of G.

Possible **Receptive Variational Principle (RVP):**

$$\tilde{h}(T) = \sup\{\tilde{h}_\mu(T) : \mu \in M(X, T)\}. \quad (1)$$

Open Problem: Is (1) always true under the above assumptions?

Variational Principle for Receptive Entropies – a few remarks

As shown in [A,D,GB,St], we have the following:

Remark 1. Assume e.g. that G is discrete amenable semigroup, $\Gamma = (N_n)$ is a Følner sequence and $h(T) > 0$. Then RVP holds. More precisely we have

$$\tilde{h}(T) = \infty = \sup\{\tilde{h}_\mu(T) : \mu \in M(X, T)\}.$$

Remark 2. Assume that X is totally disconnected. Then

$$\tilde{h}(T) \geq \sup\{\tilde{h}_\mu(T) : \mu \in M(X, T)\}.$$

However we do not know yet whether RVP holds always (under the assumptions made earlier) and we do not have counterexamples either.

References

- R.L. Adler, A.G. Konheim and M.H.McAndrew, Topological entropy, *Trans. Amer. Math. Soc.* **114** (1965), 309–319.
- A. Biś, Entropies of a semigroup of maps, *Discr. Cont. Dyn. Sys.* **11** (2004), 639-648.
- A. Biś, D. Dikranjan, A. Giordano Bruno and L. Stoyanov, *Metric entropy for group and semigroup actions*, Preprint 2018.
- R. Bowen, Entropy for group endomorphisms and homogeneous spaces, *Trans. Amer. Math. Soc.* **153** (1971), 401–414.
- R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.* **154** (1973), 125 – 136.

- J.P. Conze, Entropie d'un groupe abélien de transformations, *Z. Wahrscheinlichkeitstheorie* **25** (1972), 11–30.
- D. Dikranjan and A. Giordano Bruno, Entropy on abelian groups, *Adv. Math.* **298** (2016), 612-653.
- D. Dikranjan, M. Sanchis and S. Virili, New and old facts about entropy in uniform spaces and topological groups, *Topology and Appl.* **159** (2012), 1916 - 1942.
- D. Dikranjan, A. Fornasiero and A. Giordano Bruno, Algebraic entropy for amenable semigroup actions, Preprint 2018.
- E. I. Dinaburg, The relation between topological entropy and metric entropy, *Soviet. Math. Dokl.* **11** (1970), 13-16.

- E. Eberlein, On topological entropy of semigroups of commuting transformations, *Astérisque* **40** (1977),
- S.A. Elsanousi, A variational principle for the pressure of a continuous $\mathbb{Z}^2$ action, *Amer. J.Math.* **99** (1977), 77-100.
- E. Ghys, R. Langevin and P. Walczak, Entropie geometrique des feuilletages, *Acta Math.* **160** (1988), 105-142.
- L. W. Goodwyn, Topological entropy bounds measure-theoretic entropy, *Proc. Amer. Math. Soc.* **23** (1969), 679-688.
- K.-H. Hofmann and L. N. Stoyanov, Topological entropy of group and semigroup actions, *Adv. Math.* **115** (1995), 54-98.

- A.Kushnirenko, On metric invariants of entropy type, *Uspehi Mat. Nauk* **22** (1967), 57-66 (in Russian); English trans.: *Russ. Math. Surveys* **5** (1967), 53-61.
- M.Misiurewicz, A short proof of the variational principle for a $\mathbb{Z}_+^n$ action, *Astérisque* **40** (1977), 147-158.
- J.M. Ollagnier and D. Pinchon, The variational principle, *Studia Math.* **72** (1982), 151-159.
- D. Ruelle, Statistical mechanics on a compact set with $\mathbb{Z}^n$ action, *Trans. Amer. Math. Soc.* **185** (1973), 237-252.
- L. Todorovich, "Entropy of $\mathbb{Z}^N$ actions", Honours Thesis supervised by L. Stoyanov, University of Western Australia, 2009.
- P. Walters, "Ergodic Theory," Springer–Verlag, Berlin–Heidelberg–New York, 1980.