# Product Set Growth in Hyperbolic Geometry

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# Product sets

$G$ group, $X$ $\delta$–hyperbolic metric space, $G \curvearrowright X$

$U \subset G$ finite, $|U| =$ cardinality of $U$

$$U^n = \{g \mid g = u_1 \cdots u_n,\ u_i \in U\},\ U^n \subset G$$

# Product sets

$G = \mathbb{Z} = \langle t \rangle,\ U = \{t, 1, t^{-1}\}$

$|U^n| = 2n + 1 \leqslant n|U|$

$G = \mathbb{F}_2 = \langle a, b \rangle,\ U = \{a, b, 1, a^{-1}, b^{-1}\}$

$|U^n| = 2 \cdot 3^n - 1$ and $|U^n| > |U|^{n/2}$

# Product sets in free groups

Theorem(Razborov,Safin)

$G = \mathbb{F}_2$, $\exists \alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not cyclic,

$$|U^3| > (\alpha |U|)^2$$

# Product sets in free groups

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# Product sets in hyperbolic groups

**Theorem**

$G$ hyperbolic, $\exists \alpha > 0$ such that for all finite $U \subset G$, if $\langle U \rangle$ not virtually cyclic,

$$|U^3| > (\alpha |U|)^2 \text{ and in addition } |U^n| > (\alpha |U|)^{\lfloor (n+1)/2 \rfloor}.$$

$$\Longrightarrow h(U) = \lim_{n\to\infty} \frac{1}{n} \log |U^n| \geqslant \frac{1}{2} \log(\alpha |U|)$$

# Product sets in hyperbolic groups

$\delta > 0$, $X$ $\delta$–hyperbolic graph, $G \curvearrowright X$ properly cocompactly

- $\inf_{x \in X} d(x, gx) > 10^8\delta$ for all $g \neq 1 \in G$:

$$\alpha = \frac{1}{1200} \cdot \frac{1}{|B(x_0, 1000\delta)|^2}$$

- in general, $b := |\{g \mid d(x, gx) \leqslant 10^8\delta\}|$, and

$$\alpha = \frac{1}{10^{18}} \cdot \frac{1}{b^5} \cdot \frac{1}{|B(x_0, 1000\delta)|^2}$$

# Product sets in hyperbolic groups

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# Product sets in hyperbolic groups

$G = \mathbb{F}_2 = \langle a, b \rangle$

$U_N := \{a^{-N}, \dots, a^{-1}, 1, a, \dots, a^N\} \cup \{b\}$

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## Proposition (Button)

Let $G$ be a group. If $\exists c > 0$, $\varepsilon > 0$ such that

for all $U \subset G$ with $\langle U \rangle$ not locally virtually nilpotent

$$|U^3| > c|U|^{2+\varepsilon},$$

then $G$ has bounded exponent.

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# Groups acting on trees

## Actions on trees

free group, $A * B$, $A *_C B$

## Energy and maximal displacement of $U \subset G$

1. $E(U) := 1/|U| \sum_{u \in U} d(x, ux)$
2. $x_0 =$ minimiser for $E(U)$
3. $\lambda_0(U) := \max_{u \in U} d(x_0, ux_0)$

# Groups acting on trees

Theorem

$\forall \rho_0$, $k \geqslant \rho_0/10^{10}$, $\exists \alpha > 0$ such that

for all $G$ acting $k$-acylindrical on tree of edge-length $\rho_0$,

for all finite $U \subset G$ with $\lambda_0(U) > 10^{11}k$ and if $\langle U \rangle$ not infinite cyclic,

$$|U^3| > (\alpha|U|)^2 \text{ and in addition } |U^n| > (\alpha|U|)^{\lfloor (n+1)/2 \rfloor}.$$

# Groups acting on trees

$\rho_0$, $k \geqslant \rho_0/10^{10}$

$G$ acting $k$-acylindrical on tree of edge-length $\rho_0$

- $\alpha = \frac{\rho_0}{10^{24} k}$
- $G = A * B$, $k = \rho_0/10^{11}$. If $U$ not conjugate into vertex stabiliser, $\lambda_0(U) > 10^{11} k$.

# Groups acting on hyperbolic spaces

Theorem

$\forall \delta > 0$, $k_0 \geqslant \delta$, $n_0 > 0$, $\exists \alpha > 0$ such that

for all $G$ acting $(k_0, n_0)$-acylindrical on $X$,

for all finite $U \subset G$ with $\lambda_0(U) > \varepsilon \log(2|U|)$, if $\langle U \rangle$ not virtually cyclic,

$$|U^3| > (\frac{\alpha}{\log(2|U|)}|U|)^2 \text{ and in addition } |U^n| > (\frac{\alpha}{\log(2|U|)}|U|)^{\lfloor (n+1)/2 \rfloor}.$$

# Groups acting on hyperbolic spaces

$\delta > 0,\ k_0 \geqslant \delta,\ n_0 > 0$

$G$ acts $(k_0, n_0)$-acylindrical on $X$

- If $E(U) > \varepsilon \log(2|U|) + \delta$, then $\lambda_0(U) \geqslant \varepsilon \log(2|U|)$.
- $G = A * \mathbb{Z}, U = U_0 \cup \{t^n\}$. For large $n$: $E(U) = 1/n$, but the theorem applies.
- $\varepsilon = 10^{11} k_0,\ \alpha = \frac{1}{10^{20} n_0^5} \cdot \frac{\delta}{k_0^2}$

# Thank you for your attention!

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