MILNOR-WOLF THEOREM FOR THE GROWTH OF ENDOMORPHISMS OF LOCALLY VIRTUALLY SOLUBLE GROUPS

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Abstract. In this paper we are interested in the growth of group endomorphisms and in an analogue of the Milnor-Wolf's Theorem for the growth of finitely generated soluble groups. Indeed, we prove that if G is a locally virtually soluble group and if $\phi : G \to G$ is an endomorphism, then ϕ has either polynomial or exponential growth.

This result follows by studying the growth of automorphisms of finitely generated soluble groups, where we prove some stronger results.

1. Introduction

For a group G, denote by $\mathcal{F}(G)$ the family of all finite non-empty subsets of G. If $\phi : G \to G$ is an endomorphism and $F \in \mathcal{F}(G)$, the *growth function* of ϕ with respect to F is

$$\begin{aligned}\gamma_{\phi,F} &: \mathbb{N} \to \mathbb{N}\\ &n \mapsto |T_n(\phi, F)|,\end{aligned}$$

where

$$T_n(\phi, F) := F\phi(F)\cdots\phi^{n-1}(F) := \{f_0\phi(f_1)\cdots\phi^{n-1}(f_{n-1}) : f_0, \dots, f_{n-1} \in F\}$$

is the *n-th ϕ-trajectory* of F (see [3, 4, 6]). Here, we define $\phi^0(F) := F$ for every $F \in \mathcal{F}(G)$ and $T_0(\phi, F) := \{e_G\}$ where e_G is the identity element of G. When $e_G \in F$, we get $T_n(\phi, F) \subseteq T_{n+1}(\phi, F)$ for every $n \in \mathbb{N} \setminus \{0\}$, and hence $\{T_n(\phi, F)\}_{n\in\mathbb{N}}$ is an increasing (with respect to inclusion) family of subsets of G.

Since we want to measure the growth of the group endomorphism $\phi : G \to G$ by using the growth functions $\gamma_{\phi,F}$, we need the following equivalence relation.

Given two maps $\gamma, \gamma' : \mathbb{N} \to \{z \in \mathbb{R} : z \geq 0\}$, we write $\gamma \preceq \gamma'$ if there exists $C \in \mathbb{N}$ such that $\gamma(n) \leq \gamma'(Cn)$ for every $n \in \mathbb{N}$. Moreover, we say that γ and γ' are *equivalent*, and write $\gamma \sim \gamma'$, if $\gamma \preceq \gamma'$ and $\gamma' \preceq \gamma$; indeed, $\sim$ is an equivalence relation. Routine computations show that, for every $\alpha, \beta \in \{z \in \mathbb{R} : z \geq 0\}$, $n^\alpha \sim n^\beta$ if and only if $\alpha = \beta$; moreover, for every $a, b \in \{z \in \mathbb{R} : z > 1\}$, $a^n \sim b^n$.

A map $\gamma : \mathbb{N} \to \{z \in \mathbb{R} : z \geq 0\}$ is called

(a) *polynomial*, if $\gamma(n) \preceq n^d$ for some $d \in \mathbb{N} \setminus \{0\}$;
(b) *exponential*, if $\gamma(n) \sim e^n$;
(c) *intermediate*, if $n^d \preceq \gamma(n)$ for every $d \in \mathbb{N} \setminus \{0\}$, $\gamma(n) \preceq e^n$ and $e^n \not\preceq \gamma(n)$.

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(Our definition here slightly differs from other definitions that can be found in the literature, see for instance [14, page 4]; these small differences will not matter for the content of this paper and we prefer to work with a definition in line with some other work on growth of group endomorphisms [2, 3, 6]. See also the comment following Proposition 3.1.)

Going back to our setting, for every $F \in \mathcal{F}(G)$, we have $|F| \leq \gamma_{\phi,F}(n) \leq |F|^n$ for each $n \in \mathbb{N} \setminus \{0\}$, hence the growth of $\gamma_{\phi,F}$ is always at most exponential.

Definition 1.1 (See [2, 3, 6])**.** Let G be a group and let $\phi : G \to G$ be an endomorphism. Then:

(a) ϕ has *polynomial growth*, if $\gamma_{\phi,F}$ is polynomial for every $F \in \mathcal{F}(G)$;
(b) ϕ has *exponential growth*, if there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is exponential;
(c) ϕ has *intermediate growth*, if $\gamma_{\phi,F}$ is not exponential for every $F \in \mathcal{F}(G)$ and there exists $F_0 \in \mathcal{F}(G)$ such that γ_{ϕ,F_0} is intermediate.

We say that two group endomorphisms $\phi : G \to G$ and $\psi : H \to H$ have the *same growth type* if ϕ and ψ are both polynomial, or both exponential, or both intermediate; moreover, we say that the growth type of ϕ is *smaller than* the growth type of ψ if for every $F \in \mathcal{F}(G)$ there exists $F' \in \mathcal{F}(H)$ with $\gamma_{\phi,F} \preceq \gamma_{\psi,F'}$.

For simplicity, we say that ϕ is *subexponential* if ϕ has either polynomial or intermediate growth.

We recall also that for G a group, $\phi : G \to G$ an endomorphism and $F \in \mathcal{F}(G)$, the *algebraic entropy of ϕ with respect to F* is

$$H(\phi, F) := \lim_{n\to\infty} \frac{\log \gamma_{\phi,F}(n)}{n}.$$

Observe that this limit exists (see [5]). Now, the *algebraic entropy* of ϕ is

$$h(\phi) := \sup_{F\in\mathcal{F}(G)} H(\phi, F).$$

It was proved in [3] that $H(\phi, F) > 0$ if and only if $\gamma_{\phi,F}$ is exponential, and

$$h(\phi) > 0 \text{ if and only if } \phi \text{ has exponential growth.} \tag{1}$$

Equivalently, $h(\phi) = 0$ if and only if ϕ has subexponential growth.

The above notion of growth for group endomorphisms was inspired by the classic one. Indeed, given a finitely generated group G and a finite set of generators S of G, for every $g \in G$, denote by $\ell_S(g)$ the smallest $\ell \in \mathbb{N} \setminus \{0\}$ with

$$g = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_\ell^{\varepsilon_\ell},$$

where $s_1, \ldots, s_\ell \in S$ and $\varepsilon_1, \ldots, \varepsilon_\ell \in \{-1, 1\}$. In particular, $\ell_S(g)$ is the length of a shortest word representing g in the alphabet $S \cup S^{-1}$, where $S^{-1} := \{s^{-1} : s \in S\}$. By abuse of notation, we let $\ell_S(e_G) := 0$. The *growth function* of G with respect to S is

$$\begin{aligned} \gamma_S : \mathbb{N} &\to \mathbb{N} \\ n &\mapsto |B_S(n)|, \end{aligned}$$

where $B_S(n) := \{g \in G : \ell_S(g) \leq n\}$ is the ball of radius n in the word metric of G determined by the generating set S. Note that $B_S(0) = \{e_G\}$ and $B_S(1) = S \cup S^{-1} \cup \{e_G\}$.

Routine computations show that $\gamma_S \sim \gamma_{S'}$, for every finite generating sets S and S' of G. This observation allows us to say that G has *polynomial* (respectively, *exponential*, *intermediate*) *growth* if γ_S is polynomial (respectively, exponential, intermediate), and to notice that this definition does not depend upon S.

We mention the famous Milnor's Problem on group growth (see [15]):

(i) Are there finitely generated groups of intermediate growth?
(ii) What are the finitely generated groups of polynomial growth?

Part (i) was solved by Grigorchuk (see [8]), by constructing his famous examples of finitely generated groups with intermediate growth. Part (ii) was solved by Gromov in [10], by proving that a finitely generated group G has polynomial growth if and only if G is virtually nilpotent; in the sequel we refer to this result as Gromov's Theorem. Pioneering the work of Gromov, Milnor proved in [16] that if G is a finitely generated soluble group of subexponential growth, then G is polycyclic; later Wolf in [17] showed that a polycyclic group never has intermediate growth. As customary, we call Milnor-Wolf's Theorem the fact that a finitely generated soluble group has either polynomial or exponential growth.

The main result of this paper is a dynamic version of Milnor-Wolf's Theorem.

Theorem 1.2. *If G is a locally virtually soluble group and $\phi : G \to G$ is an endomorphism, then ϕ has either exponential or polynomial growth.*

The proof of Theorem 1.2 (see Theorem 8.5 below) is rather involved and uses the work of Gromov [10] and some ideas of Grigorchuk [9] and Milnor [16]. In our opinion the most interesting case of Theorem 1.2 is when G is finitely generated and ϕ is an automorphism, here we prove under these assumptions a stronger statement that can have independent interest (see Proposition 5.2 below):

Theorem 1.3. *Let G be a finitely generated virtually soluble group, let $\phi : G \to G$ be an automorphism and let $\langle G, \phi\rangle$ the subgroup of the holomorph $G \rtimes \mathrm{Aut}(G)$ of G generated by G and ϕ. Then either ϕ has exponential growth or $\langle G, \phi\rangle$ is virtually nilpotent. In the latter case, ϕ is polynomial.*

(For more details on the definition of $\langle G, \phi\rangle$ in Theorem 1.3, we refer to the first paragraph of Section 5.)

Both of these theorems are inspired by our preliminary investigation in [6] where we extended Gromov's Theorem and Milnor-Wolf's Theorem to arbitrary groups G, by showing that the identity automorphism id_G has polynomial growth precisely when G is locally virtually nilpotent and that if G is locally virtually soluble then id_G has either exponential or polynomial growth.

In the light of Theorem 1.2 and in the spirit of Milnor's Problem, we pose the following:

Problem 1.4 (See [6])**.** *Characterize the groups admitting no endomorphism of intermediate growth.*

We conclude this introductory section by highlighting another consequence of our work.

Theorem 1.5. *Let G be a finitely generated virtually soluble group and let $\phi : G \to G$ be an automorphism of polynomial growth. Then there exists $d \in \mathbb{N}$ (which* *depends on G and ϕ only) such that $\gamma_{\phi,F} \sim n^d$, for every finite generating set F of G.*

The number d in Theorem 1.5 can be inferred from Theorem 1.3 (see also Proposition 5.2) and the work of Grigorchuk on growth in cancellative semigroups (see [9, Theorem 2]). In fact, d can be computed with the Bass-Guivarc'h formula [1, 11] applied to the virtually nilpotent group $\langle G, \phi\rangle$.

Theorem 1.5 implies in particular that if G is a finitely generated virtually soluble group and $\phi : G \to G$ is an automorphism of polynomial growth, then $\gamma_{\phi,F} \sim \gamma_{\phi,F'}$ for every pair of finite generating sets F and F' of G. So Theorem 1.5 partially answers the following open problem.

Problem 1.6. *Let G be a finitely generated group, let $\phi : G \to G$ be an automorphism and let F and F' be any two finite generating sets of G. Is it always true that $\gamma_{\phi,F} \sim \gamma_{\phi,F'}$?*

2. Background on virtually nilpotent and virtually soluble finitely generated groups

In this section we collect useful known results on finitely generated groups, that we will frequently use. We start with the following consequence of Schreier's Lemma.

Lemma 2.1. *If G is a finitely generated group and H is a subgroup of G having finite-index, then H is finitely generated.*

It is easy to see that a finitely generated group G has the same growth type as its finite-index subgroup H, see [14, Proposition 2.5(c)].

The *lower central series* of a group G is defined inductively by $\gamma_1(G) := G$ and $\gamma_{n+1}(G) := [\gamma_n(G), G]$ for every $n \in \mathbb{N} \setminus \{0\}$. Each $\gamma_n(G)$ is a characteristic subgroup of G. The group G is *nilpotent* if $\gamma_{c+1}(G) = 1$ for some $c \in \mathbb{N}$. We say that G has *nilpotency class* c if $c \in \mathbb{N}$ is the minimum such that $\gamma_{c+1}(G) = 1$.

It is easy to verify that subgroups and quotients of nilpotent groups are nilpotent.

Lemma 2.2. *Let G be a group and let N be a normal subgroup of G. If G/N is nilpotent of nilpotency class $c \in \mathbb{N}$, then $\gamma_{c+1}(G) \leq N$.*

Proof. Arguing by induction on $n \in \mathbb{N} \setminus \{0\}$, it is easy to prove that $\gamma_n(G/N) = \gamma_n(G)N/N$. Therefore, $\gamma_{c+1}(G)N/N = \gamma_{c+1}(G/N) = 1$, and hence $\gamma_{c+1}(G) \leq N$. □

Given a group G, the *torsion* $t(G)$ of G is the set $\{g \in G : g \text{ has finite order}\}$. We recall that, when G is nilpotent, $t(G)$ is a subgroup of G, see [14, Page 31]. Moreover, a torsion finitely generated nilpotent group is finite, see [14, Proposition 2.19].

Given a property $\mathcal{P}$, the group G is said to be virtually $\mathcal{P}$ if there is a finite index subgroup $H \leq G$ such that H has property $\mathcal{P}$. In particular, G is *virtually nilpotent* if it admits a nilpotent subgroup H of finite index; equivalently, G admits a *normal* nilpotent subgroup of finite index.

Routine computations show that subgroups and quotients of virtually nilpotent groups are virtually nilpotent.

Our next lemma shows that among all finite-index nilpotent subgroups of a finitely generated virtually nilpotent group G, we may always select one that is characteristic in G.

Lemma 2.3. *Let G be a finitely generated virtually nilpotent group. Then there exists a finite-index nilpotent characteristic subgroup H of G.*

Proof. Let N be a nilpotent normal subgroup of G with $|G:N|$ finite. By Lemma 2.1, N is finitely generated. Set $k := |G:N|$ and

$$H := \langle g^k : g \in G\rangle \leq N.$$

This definition immediately gives that $\phi(H) = H$ for every automorphism ϕ of G. Moreover, as $H \leq N$, we also get that H is nilpotent. Now, the definition of H gives that N/H has exponent at most k and hence $|N:H|$ is finite because N/H is a finitely generated nilpotent group of bounded exponent, so finite. Therefore $|G:H| = |G:N||N:H|$ is finite. □

We recall that a group G is *Noetherian* if each subgroup of G is finitely generated. It is known that every finitely generated nilpotent group is Noetherian, see [14, Theorem 2.18]. We see now that also every finitely generated virtually nilpotent group is Noetherian.

Lemma 2.4. *Every subgroup H of a finitely generated virtually nilpotent group G is finitely generated.*

Proof. As G is virtually nilpotent, G contains a normal nilpotent subgroup N with $|G:N|$ finite. Then N is finitely generated by Lemma 2.1. As $H \cap N$ is a subgroup of the finitely generated nilpotent group N, we obtain that also $H \cap N$ is finitely generated.

Finally, since $|H : H \cap N| = |NH : N| \leq |G : N|$ is finite, we get that H is finitely generated. □

For a group G, the *derived series* is defined inductively by $G^{(0)} := G$ and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$ for every $n \in \mathbb{N}$. Each $G^{(n)}$ is a characteristic subgroup of G. The group G is *soluble* if $G^{(d)} = 1$ for some $d \in \mathbb{N}$. We say that G has *derived length* d if $d \in \mathbb{N}$ is the minimum such that $G^{(d)} = 1$.

Subgroups and quotients of soluble groups are soluble. Moreover, we recall that a torsion finitely generated soluble group is necessarily finite, see [13, 1.3.5].

As for virtually nilpotent groups, routine computations show that subgroups and quotients of virtually soluble groups are virtually soluble.

Lemma 2.5. *Let G be a finitely generated virtually soluble group. Then there exists a finite-index soluble characteristic subgroup H of G.*

Proof. The proof follows verbatim the proof of Lemma 2.3. □

Now we recall the following basic observation, see [16, Lemma 2].

Lemma 2.6. *Let G be a finitely generated soluble group and let A be an abelian normal subgroup of G. If the quotient G/A has a finite presentation, then there exist finitely many elements $\alpha_1, \dots, \alpha_\ell \in A$ so that every element of A can be expressed as a product of conjugates of the α_j in G.*

A group G is *polycyclic* if it has a normal series

$$1 = G_n \lhd G_{n-1} \lhd \cdots \lhd G_1 = G$$

with cyclic factor groups G_i/G_{i+1}, for each $i \in \{1, \dots, n-1\}$. We recall that a soluble group is Noetherian if and only if it is polycyclic, see [14, Proposition 2.10].

Subgroups and quotients of polycyclic groups are polycyclic. It is known that every virtually polycyclic group is finitely presented.

From [14, Theorem 2.12], a polycyclic group G contains a finite-index torsion-free normal subgroup N. Arguing as in the proof of Lemma 2.3, we obtain:

Lemma 2.7. *Let G be a polycyclic group. Then there exists a finite-index torsion-free characteristic subgroup H of G.*

3. Background on growth and entropy

In this section we first see that the growth of $\gamma_{\phi,F}$ is either bounded above by an absolute constant or at least linear. Then we recall known results and properties about the growth of group endomorphisms and the algebraic entropy, that we will use in the main part of the paper.

Proposition 3.1. *Let G be a group, let $\phi : G \to G$ be an endomorphism and let $F \in \mathcal{F}(G)$. Then one of the following holds:*
(a) there exists a constant $C > 0$ such that $\gamma_{\phi,F}(n) \leq C$ for every $n \in \mathbb{N}$,
(b) $\gamma_{\phi,F}(n) \geq n + 1$ for every $n \in \mathbb{N}$.

Proof. First observe that $\gamma_{\phi,F}$ is monotone increasing. If $\gamma_{\phi,F}$ is strictly increasing, then $\gamma_{\phi,F}(n) \geq n + 1$ for every $n \in \mathbb{N}$. In particular, we may assume that $\gamma_{\phi,F}$ is not strictly increasing, and hence there exists $n_0 \in \mathbb{N}$ such that

$$|T_{n_0}(\phi, F)| = |T_{n_0+1}(\phi, F)|.$$

Select once and for all $\bar{f} \in F$. Since $T_{n_0+1}(\phi, F) = T_{n_0}(\phi, F)\phi^{n_0}(F)$, we have

$$T_{n_0+1}(\phi, F) = T_{n_0}(\phi, F)\phi^{n_0}(\bar{f}). \tag{2}$$

We prove, by induction on $m \in \mathbb{N} \setminus \{0\}$, that

$$T_{n_0+m}(\phi, F) = T_{n_0}(\phi, F)\phi^{n_0}(\bar{f})\phi^{n_0+1}(\bar{f}) \cdots \phi^{n_0+m-1}(\bar{f}). \tag{3}$$

The case $m = 1$ is Eq. (2). Suppose that $m \geq 2$. The inductive hypothesis yields

$$\begin{aligned} T_{n_0+m}(\phi, F) &= F\phi(T_{n_0+m-1}(\phi, F)) \\ &= F\phi(T_{n_0}(\phi, F)\phi^{n_0}(\bar{f})\phi^{n_0+1}(\bar{f}) \cdots \phi^{n_0+m-2}(\bar{f})) \\ &= F\phi(T_{n_0}(\phi, F))\phi^{n_0+1}(\bar{f})\phi^{n_0+2}(\bar{f}) \cdots \phi^{n_0+m-1}(\bar{f}) \\ &= T_{n_0+1}(\phi, F)\phi^{n_0+1}(\bar{f})\phi^{n_0+2}(\bar{f}) \cdots \phi^{n_0+m-1}(\bar{f}) \\ &= T_{n_0}(\phi, F)\phi^{n_0}(\bar{f})\phi^{n_0+1}(\bar{f})\phi^{n_0+2}(\bar{f}) \cdots \phi^{n_0+m-1}(\bar{f}). \end{aligned}$$

Eq. (3) yields $\gamma_{\phi,F}(n) \leq |T_{n_0}(\phi, F)| = \gamma_{\phi,F}(n_0)$ for every $n \in \mathbb{N}$ and the lemma follows by taking $C := \gamma_{\phi,F}(n_0)$. □

Using Proposition 3.1 it is easy to show that, for the functions we are interested in in this paper (that is, the functions $\gamma_{\phi,F}$, for some group endomorphism $\phi : G \to G$ and some $F \in \mathcal{F}(G)$), our definition of $\preceq$ is equivalent to other definitions that can be found in the literature, for instance [14, page 4].

For a group G and an endomorphism $\phi : G \to G$, we say that a subgroup H of G is *ϕ-invariant* (respectively, *ϕ-stable*) if $\phi(H) \subseteq H$ (respectively, $\phi(H) = H$). Clearly, when ϕ is an automorphism, every characteristic subgroup of G is ϕ-stable. In what follows, we denote by $\phi\restriction_H$ the restriction of ϕ to the ϕ-invariant subgroup H.

The next useful observation is a direct consequence of the definitions.

Lemma 3.2. *Let G be a group, let $\phi : G \to G$ be an endomorphism and let H be a ϕ-invariant subgroup of G. Then:*

(a) the growth type of $\phi\restriction_H$ is smaller than the growth type of ϕ;
(b) if H is normal in G and $\bar{\phi} : G/H \to G/H$ is the endomorphism induced by ϕ, then the growth type of $\bar{\phi}$ is smaller than the growth type of ϕ.

The following result will be fundamental for the proof of our main theorems. Indeed, when we have a group endomorphism $\phi : G \to G$ of subexponential growth and we aim to prove that ϕ has polynomial growth, Lemma 3.3 will let us reduce to finitely generated groups.

Lemma 3.3 (See [6, Corollary 4.5])**.** *Let G be a group and let $\phi : G \to G$ be an endomorphism. If ϕ has subexponential growth (i.e., $h(\phi) = 0$), then*

$$V(\phi, F) := \langle F, \phi(F), \phi^2(F), \ldots, \phi^n(F), \ldots\rangle$$

is finitely generated for every $F \in \mathcal{F}(G)$.

It is known (e.g., see [4]) that if $\phi : G \to G$ is a group endomorphism, then $h(\phi^n) = nh(\phi)$ for every $n \in \mathbb{N} \setminus \{0\}$. This has the following consequence in view of Eq. (1).

Lemma 3.4. *Let G be a group, let $\phi : G \to G$ be an endomorphism and let n be in $\mathbb{N} \setminus \{0\}$. Then ϕ has subexponential growth if and only if ϕ^n has subexponential growth.*

In the next sections the so-called Algebraic Yuzvinski Formula will play a crucial role. Therefore, we recall this fundamental result on algebraic entropy of abelian groups.

Let $f(X)$ be a polynomial in $\mathbb{Z}[X]$ of degree $n \geq 1$. As $\mathbb{C}$ is algebraically closed, we may write

$$f(X) = s\prod_{i=1}^{n}(X - \lambda_i),$$

with $s \in \mathbb{Z}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. The *logarithmic Mahler measure* of $f(X)$ is

$$m(f(X)) := \log|s| + \sum_{\substack{i\in\{1,\ldots,n\}\\ \text{with } |\lambda_i|>1}} \log|\lambda_i|.$$

This invariant is closely related to the algebraic entropy of endomorphisms of abelian groups. Indeed, the Mahler measure of a linear transformation ϕ of a finite dimensional rational vector space $\mathbb{Q}^n$, $n \in \mathbb{N} \setminus \{0\}$, is defined as follows. Let $g(X) \in \mathbb{Q}[X]$ be the characteristic polynomial of ϕ. Then there exists a smallest $s \in \mathbb{N} \setminus \{0\}$ such that $sg(X) \in \mathbb{Z}[X]$ (so $sg(X)$ is primitive). The *Mahler measure of ϕ* is $m(\phi) := m(sg(X))$.

Theorem 3.5 (Algebraic Yuzvinski Formula, see [7])**.** *Let n be in $\mathbb{N}\setminus\{0\}$ and let $\phi:\mathbb{Q}^n\to\mathbb{Q}^n$ be an endomorphism, then $h(\phi)=m(\phi)$.*

The following lemma can be deduced from [5, Proposition 3.7]. We will use it to extend an automorphism of a finitely generated abelian group, that is, $\mathbb{Z}^n$ for some $n\in\mathbb{N}\setminus\{0\}$, to an automorphism of $\mathbb{Q}^n$ in order to apply the Algebraic Yuzvinski Formula.

Lemma 3.6. *Let G be a torsion-free abelian group, let $\phi:G\to G$ be an endomorphism and let $\phi\otimes id_{\mathbb{Q}}$ be the unique extension of ϕ to the injective hull $G\otimes_{\mathbb{Z}}\mathbb{Q}$ of G. Then $h(\phi)=h(\phi\otimes id_{\mathbb{Q}})$ and ϕ has the same growth type of $\phi\otimes id_{\mathbb{Q}}$.*

In what follows we need the following result due to Kronecker.

Theorem 3.7 (See [12])**.** *Let $f(X)\in\mathbb{Z}[X]$ be a monic polynomial with roots $\alpha_1,\dots,\alpha_k\in\mathbb{C}$. If $|\alpha_i|=1$ for every $i\in\{1,\dots,k\}$, then α_i is a root of unity for every $i\in\{1,\dots,k\}$.*

4. Reduction to automorphisms

By the next proposition, an injective endomorphism of subexponential growth is necessarily an automorphism.

Proposition 4.1. *Let G be a group and let $\phi:G\to G$ be an injective endomorphism. If ϕ is not surjective, then ϕ has exponential growth.*

Proof. Suppose that $\phi(G)<G$. Let $f\in G\setminus\phi(G)$ and set $F:=\{e_G,f\}$. We claim that, for $n\in\mathbb{N}\setminus\{0\}$, we have

$$\gamma_{\phi,F}(n)=2^n;$$

from this it immediately follows that ϕ has exponential growth. We argue by induction on n. If $n=1$, we have $\gamma_{\phi,F}(1)=|F|=2=2^1$. Assume now that $\gamma_{\phi,F}(n)=2^n$. Observe that this implies that the 2^n many products

$$e_1\phi(e_2)\cdots\phi^{n-1}(e_n)\quad(\text{for } e_1,\dots,e_n\in F)$$

are all distinct. Let $e_1,\dots,e_{n+1},e'_1,\dots,e'_{n+1}\in F$ and suppose that

$$e_1\phi(e_2)\cdots\phi^{n-1}(e_n)\phi^n(e_{n+1})=e'_1\phi(e'_2)\cdots\phi^{n-1}(e'_n)\phi^n(e'_{n+1}). \tag{4}$$

Multiplying both sides of this equation by the subgroup $\phi(G)$, we get

$$e_1\phi(G)=e'_1\phi(G).$$

As $e_1,e'_1\in F=\{e_G,f\}$ and $f\notin\phi(G)$, we get $e_1=e'_1$. Therefore, by simplifying $e_1=e'_1$ on both sides of Eq. (4), we deduce

$$\begin{aligned}\phi(e_2\phi(e_3)\cdots\phi^{n-2}(e_n)\phi^{n-1}(e_{n+1})) &= \phi(e_2)\cdots\phi^{n-1}(e_n)\phi^n(e_{n+1})\\ &= \phi(e'_2)\cdots\phi^{n-1}(e'_n)\phi^n(e'_{n+1})\\ &= \phi(e'_2\phi(e'_3)\cdots\phi^{n-2}(e'_n)\phi^{n-1}(e'_{n+1})).\end{aligned}$$

Since ϕ is injective, we obtain

$$e_2\phi(e_3)\cdots\phi^{n-2}(e_n)\phi^{n-1}(e_{n+1})=e'_2\phi(e'_3)\cdots\phi^{n-2}(e'_n)\phi^{n-1}(e'_{n+1})$$

and the inductive hypothesis gives $e_i=e'_i$ for every $i\in\{2,\dots,n+1\}$. Thus $\gamma_{\phi,F}(n+1)=2^{n+1}$. □

We see now that the group endomorphism $\phi:G\to G$ has the same growth type of the endomorphism $\bar\phi:G/\mathrm{Ker}(\phi)\to G/\mathrm{Ker}(\phi)$ induced by ϕ.

Lemma 4.2. *Let G be a group, let $\phi : G \to G$ be an endomorphism and let $\bar{\phi} : G/\mathrm{Ker}(\phi) \to G/\mathrm{Ker}(\phi)$ be the endomorphism induced by ϕ. Let $F \in \mathcal{F}(G)$ and let $\bar{F}$ be the projection of F on $G/\mathrm{Ker}(\phi)$. Then*

$$\gamma_{\phi,F}(n) \leq |F|\gamma_{\bar{\phi},\bar{F}}(n-1) \leq |F|\gamma_{\phi,F}(n-1) \tag{5}$$

for each $n \in \mathbb{N} \setminus \{0\}$. In particular, $\gamma_{\phi,F} \sim \gamma_{\bar{\phi},\bar{F}}$, $H(\phi, F) = H(\bar{\phi}, \bar{F})$ and $h(\phi) = h(\bar{\phi})$.

Proof. Set $K := \mathrm{Ker}(\phi)$. We denote by $^{-} : G \to G/K$ the natural projection of G onto G/K, and we use the usual "bar" notation.

Let S be a subset of G. We claim that

$$|\phi(S)| = |\bar{S}|. \tag{6}$$

Indeed, for $x, y \in S$, we have $\phi(x) = \phi(y)$ if and only if $xy^{-1} \in K$; this in turn happens if and only if $\overline{xy^{-1}} = 1$, that is, $\bar{x} = \bar{y}$.

Now, let $F \in \mathcal{F}(G)$ and $n \in \mathbb{N}$. From Eq. (6) applied with S replaced by $T_n(\phi, F)$, we get

$$|\phi(T_n(\phi, F))| = |\overline{T_n(\phi, F)}| = |T_n(\bar{\phi}, \bar{F})|. \tag{7}$$

From Eq. (7) the first part of the lemma immediately follows. In fact, given $n \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} \gamma_{\phi,F}(n) &= |T_n(\phi, F)| = |F\phi(T_{n-1}(\phi, F))| \leq |F||\phi(T_{n-1}(\phi, F))| \\ &= |F||T_{n-1}(\bar{\phi}, \bar{F})| = |F|\gamma_{\bar{\phi},\bar{F}}(n-1) \leq |F|\gamma_{\phi,F}(n-1). \end{aligned}$$

From these inequalities, we see that if $\gamma_{\bar{\phi},\bar{F}}$ (respectively, $\gamma_{\phi,F}$) is bounded above by an absolute constant, then so is $\gamma_{\phi,F}$ (respectively, $\gamma_{\bar{\phi},\bar{F}}$). In particular, $\gamma_{\phi,F} \sim \gamma_{\bar{\phi},\bar{F}}$. Suppose that $\gamma_{\phi,F}$ and $\gamma_{\bar{\phi},\bar{F}}$ are not bounded above by an absolute constant. Then Proposition 3.1 yields $\gamma_{\bar{\phi},\bar{F}}(n) \geq n + 1$, for every $n \in \mathbb{N}$. Now, a moment's thought yields $|F|\gamma_{\bar{\phi},\bar{F}}(n-1) \leq \gamma_{\bar{\phi},\bar{F}}(Cn)$ for every $n \in \mathbb{N} \setminus \{0\}$, for some absolute constant $C > 0$. Thus $\gamma_{\phi,F}(n) \leq |F|\gamma_{\bar{\phi},\bar{F}}(n-1) \leq \gamma_{\bar{\phi},\bar{F}}(Cn)$ for every $n \in \mathbb{N} \setminus \{0\}$ and hence $\gamma_{\phi,F} \preceq \gamma_{\bar{\phi},\bar{F}}$. As it is clear that $\gamma_{\bar{\phi},\bar{F}} \preceq \gamma_{\phi,F}$, we get $\gamma_{\phi,F} \sim \gamma_{\bar{\phi},\bar{F}}$.

From Eq. (5), we obtain

$$H(\phi, F) = \lim_{n\to\infty} \frac{\log \gamma_{\phi,F}(n)}{n} = \lim_{n\to\infty} \frac{\log \gamma_{\bar{\phi},\bar{F}}(n)}{n} = H(\bar{\phi}, \bar{F})$$

and hence $h(\phi) = h(\bar{\phi})$. □

For a group G and an endomorphism $\phi : G \to G$, let

$$\mathrm{Ker}_\infty(\phi) := \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathrm{Ker}(\phi^n). \tag{8}$$

It is straightforward to verify that $\phi(\mathrm{Ker}_\infty(\phi)) \subseteq \mathrm{Ker}_\infty(\phi)$ and that $\phi^{-1}(\mathrm{Ker}_\infty(\phi)) = \mathrm{Ker}_\infty(\phi)$. Moreover, the induced endomorphism $\bar{\phi} : G/\mathrm{Ker}_\infty(\phi) \to G/\mathrm{Ker}_\infty(\phi)$ is injective.

Lemma 4.3. *Let G be a group, let $\phi : G \to G$ be an endomorphism and let $\bar{\phi} : G/\mathrm{Ker}_\infty(\phi) \to G/\mathrm{Ker}_\infty(\phi)$ be the endomorphism induced by ϕ. Assume there exists $n_0 \in \mathbb{N} \setminus \{0\}$ with $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$. Let $F \in \mathcal{F}(G)$ and let $\bar{F}$ be the projection of F on $G/\mathrm{Ker}_\infty(\phi)$. Then*

$$\gamma_{\phi,F}(n) \leq |F|^{n_0}\gamma_{\bar{\phi},\bar{F}}(n - n_0) \leq |F|^{n_0}\gamma_{\phi,F}(n - n_0),$$

for every $n \in \mathbb{N}$ with $n \geq n_0$. In particular, $\gamma_{\phi,F} \sim \gamma_{\bar\phi,\bar F}$, $H(\phi,F) = H(\bar\phi,\bar F)$ and $h(\phi) = h(\bar\phi)$.

Proof. Set $K := \mathrm{Ker}_\infty(\phi)$. For $n \in \mathbb{N} \setminus \{0\}$, let $K_n := \mathrm{Ker}(\phi^n)$ and denote by $\bar\phi_n : G/K_n \to G/K_n$ the endomorphism induced by ϕ on G/K_n and by $\pi_n : G \to G/K_n$ the natural projection. Applying Lemma 4.2 inductively, for every $n \in \mathbb{N}$ with $n \geq n_0$, we get

$$\begin{aligned}\gamma_{\phi,F}(n) &\leq |F|\gamma_{\bar\phi_1,\pi_1(F)}(n-1)\\ &\leq |F|(|F|\gamma_{\bar\phi_2,\pi_2(F)}(n-2)) \leq \cdots \leq |F|^{n_0}\gamma_{\bar\phi_{n_0},\pi_{n_0}(F)}(n-n_0).\end{aligned}$$

As $K = \mathrm{Ker}(\phi^{n_0})$, we have $\pi_{n_0}(F) = \bar F$ and $\bar\phi_{n_0} = \bar\phi$ and hence

$$\gamma_{\phi,F}(n) \leq |F|^{n_0}\gamma_{\bar\phi,\bar F}(n-n_0).$$

The inequality $\gamma_{\bar\phi,\bar F}(n-n_0) \leq \gamma_{\phi,F}(n-n_0)$ is clear.

The rest of the proof follows verbatim the proof of Lemma 4.2. □

Lemma 4.4. *Let G be a group and let $\phi : G \to G$ be an endomorphism. If $\mathrm{Ker}_\infty(\phi)$ is finitely generated, then $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$ for some $n_0 \in \mathbb{N} \setminus \{0\}$.*

Proof. Let $k_1, \ldots, k_\ell$ be a family of generators for $\mathrm{Ker}_\infty(\phi)$. From the definition of $\mathrm{Ker}_\infty(\phi)$ in Eq. (8), for every $i \in \{1, \ldots, \ell\}$, there exists $t_i \in \mathbb{N} \setminus \{0\}$ with $s_i \in \mathrm{Ker}(\phi^{t_i})$. Now, consider $n_0 := \max(t_i : i \in \{1, \ldots, \ell\})$ and observe that $k_1, \ldots, k_\ell \in \mathrm{Ker}(\phi^{n_0})$. Therefore

$$\mathrm{Ker}(\phi^{n_0}) \leq \mathrm{Ker}_\infty(\phi) = \langle k_1, \ldots, k_\ell\rangle \leq \mathrm{Ker}(\phi^{n_0})$$

and hence $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$. □

Lemma 4.5. *Let G be a Noetherian group and let $\phi : G \to G$ be an endomorphism. Then $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$ for some $n_0 \in \mathbb{N} \setminus \{0\}$.*

Proof. Since G is Noetherian, $\mathrm{Ker}_\infty(\phi)$ is finitely generated and hence $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$ for some $n_0 \in \mathbb{N} \setminus \{0\}$ by Lemma 4.4. □

Corollary 4.6. *Let G be a finitely generated virtually nilpotent group and let $\phi : G \to G$ be an endomorphism. Then $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$ for some $n_0 \in \mathbb{N} \setminus \{0\}$.*

Proof. By Lemma 2.4, G is Noetherian, so Lemma 4.5 applies. □

The following lemma is more general than Corollary 4.6. On the other hand, it does not cover Lemma 4.5 because there exist Noetherian groups that are not finitely presented. (For instance, a Tarski monster is not finitely presented and is clearly Noetherian because the only proper non-identity subgroups are cyclic of prime order.)

Lemma 4.7. *Let G be a finitely generated group, let $\phi : G \to G$ be an endomorphism, and assume that $G/\mathrm{Ker}_\infty(\phi)$ is finitely presented. Then $\mathrm{Ker}_\infty(\phi) = \mathrm{Ker}(\phi^{n_0})$ for some $n_0 \in \mathbb{N} \setminus \{0\}$.*

Proof. Let $K := \mathrm{Ker}_\infty(\phi)$ and $K_n := \mathrm{Ker}(\phi^n)$ for every $n \in \mathbb{N} \setminus \{0\}$. We denote by $\bar{\ } : G \to G/K$ the natural projection of G onto $G/K = \bar G$. Let $g_1, \ldots, g_\kappa$ be a finite set of generators for G. Observe that $\bar g_1, \ldots, \bar g_\kappa$ is a finite set of generators for $\bar G$. As $\bar G$ is finitely presented, $\bar G$ has a finite presentation (in the generators $\bar g_1, \ldots, \bar g_\kappa$):

$$\bar G = \langle x_1, \ldots, x_\kappa : r_1(x_1, \ldots, x_\kappa), \ldots, r_\ell(x_1, \ldots, x_\kappa)\rangle.$$

For $i \in \{1, \ldots, \ell\}$, let $k_i := r_i(g_1, \ldots, g_\kappa)$ and observe that $k_i \in K$ because

$$\bar{k}_i = r_i(\bar{g}_1, \ldots, \bar{g}_\kappa) = 1.$$

We claim that

$$K = \langle k_i^g : i \in \{1, \ldots, \ell\}, g \in G \rangle.$$

Let us denote by H the group $\langle k_i^g : i \in \{1, \ldots, \ell\}, g \in G \rangle$. As $k_i \in K$ and $K \trianglelefteq G$, we have $H \leq K$. We prove the reverse inclusion. Let $k \in K$. Since G is generated by $g_1, \ldots, g_\kappa$, the element k must be written as a word in $g_1, \ldots, g_\kappa$, that is, $k = w(g_1, \ldots, g_\kappa)$ for some word $w(x_1, \ldots, x_\kappa)$. Now,

$$e_{\bar{G}} = \bar{k} = \overline{w(g_1, \ldots, g_\kappa)} = w(\bar{g}_1, \ldots, \bar{g}_\kappa)$$

and hence, directly from the definition of group-presentation, the word $w(x_1, \ldots, x_\kappa)$ lies in the normal closure

$$\langle r_i(x_1, \ldots, x_\kappa)^x : i \in \{1, \ldots, \ell\}, x \in \langle x_1, \ldots, x_\kappa \rangle \rangle.$$

Therefore

$$\begin{aligned} k = w(g_1, \ldots, g_\kappa) &\in \langle r_i(g_1, \ldots, g_\kappa)^g : i \in \{1, \ldots, \ell\}, g \in G \rangle \\ &= \langle k_i^g : i \in \{1, \ldots, \ell\}, g \in G \rangle = H. \end{aligned}$$

Observe now that K is the union of the infinite chain $K_1 \leq K_2 \leq K_3 \leq \cdots$. Let $n_0 \in \mathbb{N} \setminus \{0\}$ with $k_1, \ldots, k_\ell \in K_{n_0}$. Observe that n_0 exists because $k_1, \ldots, k_\ell$ is a finite set in K. Since $K_{n_0} \trianglelefteq G$, we now get

$$K_{n_0} \leq K = \langle k_i^g : i \in \{1, \ldots, \ell\}, g \in G \rangle \leq K_{n_0},$$

that is, $K = K_{n_0}$. □

5. Classic growth versus endomorphism growth

Let G be a finitely generated group and let $\phi : G \to G$ be an automorphism. We consider the semidirect product $G \rtimes \langle \phi \rangle$ given by the action of ϕ on G, and we identify G with the subgroup $G \times \{id_G\}$ of $G \rtimes \langle \phi \rangle$. Using this identification, we will write $\langle G, \phi \rangle$ in place of $G \rtimes \langle \phi \rangle$. Clearly, $\langle G, \phi \rangle$ is a finitely generated group.

(For not making the notation cumbersome, if N is a normal ϕ-stable subgroup of G, we write simply $\langle N, \phi \rangle$ and $\langle G/N, \phi \rangle$ in place of, respectively, $\langle N, \phi \restriction_N \rangle$ and $\langle G/N, \bar{\phi} \rangle$, where $\bar{\phi} : G/N \to G/N$ is the automorphism induced by ϕ; hopefully this abuse of notation will not cause any confusion.)

In this section we give relations between the growth of the group automorphism $\phi : G \to G$ and the classic growth of the finitely generated group $\langle G, \phi \rangle$.

Lemma 5.1. *Let G be a finitely generated group, let $\phi : G \to G$ be an automorphism, let F be in $\mathcal{F}(G)$ and let n be in $\mathbb{N} \setminus \{0\}$. Then $\gamma_{\phi,F}(n) = |(F\phi^{-1})^n|$.*

Proof. Computing in $\langle G, \phi \rangle$, we see that for $g \in G$, we have $\phi(g) = \phi^{-1} g \phi$. Then

$$\begin{aligned} T_n(\phi, F) &= F\phi(F)\phi^2(F) \cdots \phi^{n-1}(F) = F(\phi^{-1}F\phi)(\phi^{-2}F\phi^2) \cdots (\phi^{-(n-1)}F\phi^{n-1}) \\ &= (F\phi^{-1})(F\phi^{-1}) \cdots (F\phi^{-1})\phi^n = (F\phi^{-1})^n \phi^n. \end{aligned}$$

Therefore, $\gamma_{\phi,F}(n) = |T_n(\phi, F)| = |(F\phi^{-1})^n|$. □

Using one of our favourite results of Grigorchuk [9], we now prove that ϕ has polynomial growth precisely when $\langle G, \phi\rangle$ has polynomial growth. By Gromov's Theorem the latter condition is equivalent to require that $\langle G, \phi\rangle$ is virtually nilpotent. (We thank Grigorchuk for sharing with us the ideas in [9].)

Proposition 5.2. *Let G be a finitely generated group and let $\phi : G \to G$ be an automorphism. Then the following conditions are equivalent:*

(a) ϕ has polynomial growth;
(b) $\langle G, \phi\rangle$ has polynomial growth;
(c) $\langle G, \phi\rangle$ is virtually nilpotent.

Proof. (a)⇒(b) Let F be a finite set of generators for G with $e_G \in F$ and consider

$$S := \bigcup_{n\in\mathbb{N}} (F\phi^{-1})^n,$$

where the computations are performed in $\langle G, \phi\rangle$. Then, by construction, S contains e_G and is closed by taking products, therefore S is the subsemigroup of G with identity generated by $F\phi^{-1}$. Observe that S is a cancellative semigroup, because $S \subseteq G$ and G is a group. By hypothesis $\gamma_{\phi,F}(n)$ is polynomial, so the function

$$n \mapsto |(F\phi^{-1})^n|$$

is polynomial by Lemma 5.1. Since S is a cancellative semigroup of polynomial growth, S has the group of left quotients $S^{-1}S$ by [9, Corollary 1]. Clearly, $S^{-1}S = \langle G, \phi\rangle$, since $\langle G, \phi\rangle$ is generated by $F\phi^{-1}$ as a group, and in particular $\langle G, \phi\rangle = \langle S\rangle$. Now [9, Theorems 1 and 2] show that the polynomial growth of the semigroup S forces a polynomial growth of the group $S^{-1}S = \langle G, \phi\rangle$.

(b)⇒(a) By hypothesis, there exist $d_1, d_2 \in \mathbb{N}$ such that, for each inverse-closed finite subset S of $\langle G, \phi\rangle$, we have

$$|S^n| \le d_1 n^{d_2},$$

for every $n \in \mathbb{N}$. In particular, if F is a finite subset of G, by Lemma 5.1 we get

$$\gamma_{\phi,F}(n) = |(F\phi^{-1})^n| \le |(F\phi^{-1} \cup (F\phi^{-1})^{-1})^n| \le d_1 n^{d_2},$$

thus ϕ has polynomial growth.

(b)⇔(c) This is Gromov's Theorem. □

To apply the ideas in Proposition 5.2 more directly in later arguments, we prove the following result.

Proposition 5.3. *Let G be a finitely generated group and let $\phi : G \to G$ be an automorphism. If $\ell \in \mathbb{N} \setminus \{0\}$ and N is a finite-index normal ϕ-stable subgroup of G, then $\langle N, \phi^\ell\rangle$ has finite index in $\langle G, \phi\rangle$. Consequently:*

(a) $\langle G, \phi\rangle$ is virtually nilpotent if and only if $\langle N, \phi^\ell\rangle$ is virtually nilpotent;
(b) $\langle G, \phi\rangle$ has polynomial growth if and only if $\langle N, \phi^\ell\rangle$ has polynomial growth.

Proof. Let N be a finite-index normal ϕ-stable subgroup of G. Observe that a set of representatives for the right cosets of N in G is also a set of representatives for the right cosets of $\langle N, \phi\rangle$ in $\langle G, \phi\rangle$. Therefore $|\langle G, \phi\rangle : \langle N, \phi\rangle| = |G : N|$. Moreover, $|\langle N, \phi\rangle : \langle N, \phi^\ell\rangle| = \ell$. Hence, $\langle N, \phi^\ell\rangle$ is a finite-index subgroup of $\langle G, \phi\rangle$.

Consequently, the statement in item (a) holds true. Moreover, (b) follows from (a) and Proposition 5.2. □

It is rather hard for the authors to skip to the next section without adding a remark that we believe is pivotal to have a better understanding on growth of automorphisms. In fact, we believe that Proposition 5.2 is only the tip of an iceberg and dare to make the following conjecture.

Conjecture 5.4. *Let G be a finitely generated group and let $\phi : G \to G$ be an automorphism. Then ϕ has exponential growth if and only if $\langle G, \phi\rangle$ has exponential growth.*

Clearly, if ϕ has exponential growth, so does $\langle G, \phi\rangle$.

6. Automorphisms acting nilpotently

Let G be a group and let $\phi : G \to G$ be an automorphism. For $n \in \mathbb{N} \setminus \{0\}$, we define the symbol $[G, \underbrace{\phi, \dots, \phi}_{n\,\text{times}}]$ (or, $[G,_n \phi]$ for short) inductively:

$$[G,_1 \phi] := [G, \phi] := \langle g^{-1}\phi(g) : g \in G\rangle \quad \text{and} \quad [G,_{n+1} \phi] := [[G,_n \phi], \phi] \text{ for } n \geq 1.$$

Definition 6.1. Let G be a group and let $\phi : G \to G$ be an automorphism. We say that ϕ *acts nilpotently* on G if there exists $n \in \mathbb{N} \setminus \{0\}$ such that $[G,_n \phi] = 1$.

We start with basic properties, whose proofs are clear from the definition.

Lemma 6.2. *Let G be a group, let $\phi : G \to G$ be an automorphism and let H be a ϕ-stable subgroup of G. If ϕ acts nilpotently on G, then ϕ acts nilpotently on H.*

Proof. The thesis follows from the fact that $[H,_n \phi] \subseteq [G,_n, \phi]$ for every $n \in \mathbb{N} \setminus \{0\}$. □

Lemma 6.3. *Let G be a group and let $\phi : G \to G$ be an automorphism. If ϕ acts nilpotently on G, then ϕ^ℓ acts nilpotently on G, for every $\ell \in \mathbb{N} \setminus \{0\}$.*

Proof. The thesis follows from the fact that $[G,_n \phi^\ell] \subseteq [G,_n \phi]$ for every $n \in \mathbb{N} \setminus \{0\}$. □

Proposition 6.4. *Let G be a finitely generated group and let $\phi : G \to G$ be an automorphism. Then the following conditions are equivalent:*

(a) $\langle G, \phi\rangle$ is nilpotent;
(b) G is nilpotent and ϕ acts nilpotently on G.

Proof. (a)⇒(b) Let $H := \langle G, \phi\rangle$ and assume that H is nilpotent, that is, $\gamma_{d+1}(H) = 1$, for some $d \in \mathbb{N} \setminus \{0\}$. Clearly, G is nilpotent; moreover, ϕ acts nilpotently on G since $[G,_d \phi] \leq \gamma_{d+1}(H) = 1$.

(b)⇒(a) Let $c, m \in \mathbb{N} \setminus \{0\}$ where c is the nilpotency class of G and $[G,_m \phi] = 1$. We show, by induction on c, that $\gamma_{mc+1}(\langle G, \phi\rangle) = 1$. Suppose that $c = 1$, that is, G is abelian. As $\langle G, \phi\rangle/G$ is cyclic and $[G, G] = 1$, we have

$$\gamma_2(\langle G, \phi\rangle) = [\langle G, \phi\rangle, \langle G, \phi\rangle] = [G, \langle G, \phi\rangle] = [G, \phi]$$

and, arguing inductively on $m > 2$,

$$\gamma_{m+1}(\langle G, \phi\rangle) = [\gamma_m(\langle G, \phi\rangle), \langle G, \phi\rangle] = [[G,_{m-1} \phi], \phi] = [G,_m \phi] = 1.$$

Suppose now that $c > 1$. Let $\bar{G} := G/\gamma_c(G)$ and let $\bar{\phi} : \bar{G} \to \bar{G}$ be the automorphism induced by ϕ on $\bar{G}$. As $[\bar{G},_m \bar{\phi}] = 1$, the inductive hypothesis yields $\gamma_{m(c-1)+1}(\langle \bar{G}, \bar{\phi}\rangle) = 1$, that is, $\gamma_{m(c-1)+1}(\langle G, \phi\rangle) \leq \gamma_c(G)$ by Lemma 2.2. Therefore

$$\gamma_{mc+1}(\langle G,\phi\rangle) = [\gamma_{m(c-1)+1}(\langle G,\phi\rangle), \underbrace{\langle G,\phi\rangle, \dots, \langle G,\phi\rangle}_{m \text{ times}}] \leq [\gamma_c(G), \underbrace{\langle G,\phi\rangle, \dots, \langle G,\phi\rangle}_{m \text{ times}}].$$

Since $\gamma_c(G)$ is a central subgroup of G, we get $[\gamma_c(G), \langle G, \phi\rangle] = [\gamma_c(G), \phi]$ and hence

$$\gamma_{mc+1}(\langle G,\phi\rangle) \leq [\gamma_c(G), \underbrace{\phi, \dots, \phi}_{m \text{ times}}] = [\gamma_c(G),_m \phi] = 1.$$

Thus, $\langle G, \phi\rangle$ is nilpotent. □

For abelian groups we have the following clear set-theoretic description of the subgroups $[G,_n \phi]$.

Lemma 6.5. *Let G be an abelian group and let $\phi : G \to G$ be an automorphism. Then, for every $n \in \mathbb{N} \setminus \{0\}$,*

$$[G,_n \phi] = (\phi - id_G)^n(G) = \{(\phi - id_G)^n(g) : g \in G\}. \tag{9}$$

Consequently, $[G,_n \phi] = 1$ if and only if $(\phi - id_G)^n = 0$.

Proof. We proceed by induction on $n \in \mathbb{N} \setminus \{0\}$. (We use an additive notation for G.) When $n = 1$, we have

$$\begin{aligned} [G,\phi] &= \langle [g,\phi] : g \in G\rangle = \langle -g + \phi^{-1} g \phi : g \in G\rangle \\ &= \langle -g + \phi(g) : g \in G\rangle = \langle (\phi - id_G)(g) : g \in G\rangle = \{(\phi - id_G)(g) : g \in G\}. \end{aligned}$$

Let now $n \in \mathbb{N} \setminus \{0\}$; then

$$\begin{aligned} [G,_{n+1} \phi] &= [[G,_n \phi], \phi] = [\{(\phi - id_G)^n(g) : g \in G\}, \phi] \\ &= \{(\phi - id_G)(\phi - id_G)^n(g) : g \in G\} = \{(\phi - id_G)^{n+1}(g) : g \in G\}. \end{aligned}$$

This proves Eq. (9). The second assertion is an immediate consequence. □

The next result will be used in Proposition 6.7.

Lemma 6.6. *Let G be a finitely generated abelian group and let $\phi : G \to G$ be an automorphism such that ϕ acts nilpotently on G. Then there exists a sequence*

$$G = C_0 > C_1 > \cdots > C_m = 0$$

such that, for each $i \in \{0, \dots, m-1\}$,

$$C_i/C_{i+1} \text{ is cyclic} \quad \text{and} \quad [C_i, \phi] \leq C_{i+1}.$$

Proof. From Proposition 6.4, $\langle G, \phi\rangle$ is nilpotent and hence polycyclic. Since each $\gamma_i(G)/\gamma_{i+1}(G)$ is finitely generated and $\gamma_2(G) \leq G$ by Lemma 6.5, we may take $(C_i)_{i=0}^m$ to be any normal series of $\langle G, \phi\rangle$ passing through G, witnessing that $\langle G, \phi\rangle$ is polycyclic and refining the lower central series of $\langle G, \phi\rangle$. □

The next result will be applied in the proof of Theorem 8.2.

Proposition 6.7. *Let G be a polycyclic group of derived length $d \in \mathbb{N} \setminus \{0\}$ and let $\phi : G \to G$ be an automorphism such that ϕ acts nilpotently on $G^{(i)}/G^{(i+1)}$ for every $i \in \{1, \dots, d-1\}$. Then there exists a normal series*

$$G = G_1 > G_2 > \cdots > G_{\kappa-1} > G_\kappa = 1,$$

- *refining the derived series of G;*
- *with G_i/G_{i+1} cyclic for each $i \in \{1, \dots, \kappa - 1\}$;*
- *with $[G_i, \phi] \leq G_{i+1}$ for each $i \in \{1, \dots, \kappa - 1\}$.*

Proof. Let $i \in \{1, \dots, d-1\}$. Then $G^{(i)}/G^{(i+1)}$ is abelian and finitely generated since G is polycyclic. By Lemma 6.6 applied with $B := G^{(i)}/G^{(i+1)}$, there exists a normal series

$$G^{(i)} = C_0 > C_1 > \cdots > C_{m_i} = G^{(i+1)}$$

such that C_j/C_{j+1} is cyclic and $[C_j, \phi] \leq C_{j+1}$ for every $j \in \{0, \dots, m_i - 1\}$. □

We give now an auxiliary lemma used in the proof of Lemma 8.4

Lemma 6.8. *Let A be an infinite finitely generated torsion-free abelian group and let $\phi : A \to A$ be an automorphism such that ϕ acts nilpotently on A. Then there exists a ϕ-stable subgroup B of A such that A/B is infinite cyclic (i.e., isomorphic to $\mathbb{Z}$).*

Proof. If $A/[A, \phi]$ is finite, then, arguing by induction on $n \in \mathbb{N} \setminus \{0\}$ and using the set-theoretic description of $[A,_n \phi]$ in Lemma 6.5, one can prove that $[A,_n \phi]/[A,_{n+1} \phi]$ is finite for every $n \in \mathbb{N} \setminus \{0\}$. Since ϕ acts nilpotently on A, there exists $n_0 \in \mathbb{N} \setminus \{0\}$ with $[A,_{n_0}, \phi] = 1$, however this would mean that A is finite, contradicting the fact that A is infinite by hypothesis.

Therefore $A/[A, \phi]$ is an infinite finitely generated abelian group, so there exists a subgroup B of A such that $[A, \phi] \leq B$ and A/B is infinite cyclic. Since B contains $[A, \phi]$, B is ϕ-stable. □

7. Growth of endomorphisms of locally virtually nilpotent groups

In this section we prove that, if G is a locally virtually nilpotent group, then every endomorphism of G has either polynomial or exponential growth.

We start with a technical lemma which permits (for example) to restrict to torsion-free finitely generated nilpotent (or abelian) groups.

Lemma 7.1. *Let G be a finitely generated nilpotent group, let $\phi : G \to G$ be an automorphism and let T be a finite normal ϕ-stable subgroup of G. If there exists $\ell \in \mathbb{N} \setminus \{0\}$ such that $\langle G/T, \phi^\ell \rangle$ is nilpotent, then there exists $\ell' \in \mathbb{N} \setminus \{0\}$ such that $\langle G, \phi^{\ell'} \rangle$ is nilpotent.*

Proof. Assume that G has nilpotency class c and that $\langle G/T, \phi^\ell \rangle$ has nilpotency class d. Since T is finite and $\phi^\ell \restriction_T : T \to T$ is an automorphism, there exists a non-zero multiple ℓ' of ℓ such that $\phi^{\ell'} \restriction_T = id_T$ (for instance, we may take $\ell' := \ell|\mathrm{Aut}(T)|$). We prove that $L := \langle G, \phi^{\ell'} \rangle$ is nilpotent. By Lemma 2.2 applied to $\langle G, \phi^\ell \rangle$ and to its normal subgroup T, we have $\gamma_{d+1}(\langle G, \phi^\ell \rangle) \leq T$ and hence

$$\gamma_{d+1}(L) \leq \gamma_{d+1}(\langle G, \phi^\ell \rangle) \leq T.$$

Therefore, since $\phi^{\ell'}$ centralizes T,

$$\begin{aligned}\gamma_{d+c+1}(L) &= [\gamma_{d+1}(L), \underbrace{L, \dots, L}_{c\,\text{times}}] \\ &\leq [T, \underbrace{L, \dots, L}_{c\,\text{times}}] \\ &\leq [T, \underbrace{G, \dots, G}_{c\,\text{times}}] \leq [\underbrace{G, \dots, G}_{c+1\,\text{times}}] = \gamma_{c+1}(G) = 1.\end{aligned}$$

Thus, L is nilpotent of nilpotency class at most $c + d$. $\square$

The following lemma will be fundamental for this and for the next section. It uses the Algebraic Yuzvinski Formula.

Lemma 7.2. *Let G be a finitely generated abelian group and let $\phi : G \to G$ be an automorphism of subexponential growth. Then there exists $\ell \in \mathbb{N} \setminus \{0\}$ such that ϕ^ℓ acts nilpotently on G and consequently $\langle G, \phi^\ell \rangle$ is nilpotent.*

Proof. Since the torsion $t(G)$ is a finite ϕ-stable subgroup of G, by Lemma 7.1 we can assume without loss of generality that G is torsion-free, that is, $G \cong \mathbb{Z}^m$ for some $m \in \mathbb{N} \setminus \{0\}$.

Let $p_\phi(X)$ be the characteristic polynomial of $\phi \otimes \mathbb{Q}$. As ϕ is an automorphism of $G \cong \mathbb{Z}^m$, we see that $p_\phi(X) \in \mathbb{Z}[X]$ and that $p_\phi(X)$ is monic. As $h(\phi \otimes \mathbb{Q}) = h(\phi) = 0$ in view of Eq. (1) and Lemma 3.6, by Theorem 3.5 we get that $|\lambda| \leq 1$ for each eigenvalue λ of $\phi \otimes \mathbb{Q}$. Recall that the coefficient of degree zero of $p_\phi(X) \in \mathbb{Z}[X]$ is (up to a sign change) the product of the eigenvalues of $\phi \otimes \mathbb{Q}$. Consequently, $|\lambda| = 1$ for each eigenvalue λ of $\phi \otimes \mathbb{Q}$, so Theorem 3.7 yields that each eigenvalue of $\phi \otimes \mathbb{Q}$ is a root of unity. Thus

$$p_\phi(X) = \prod_{i=1}^{t} (X - \omega_i)^{m_i},$$

where $m_1, \dots, m_t \in \mathbb{N} \setminus \{0\}$, $m = m_1 + \cdots + m_t$, and $\omega_1, \dots, \omega_t \in \mathbb{C}$ are roots of unity. Let ℓ be the least common multiple of the order of the roots of unity $\omega_1, \dots, \omega_t$. Now,

$$p_{\phi^\ell}(X) = \prod_{i=1}^{t} (X - \omega_i^\ell)^{m_i} = \prod_{i=1}^{t} (X - 1)^{m_i} = (X - 1)^{\sum_{i=1}^{t} m_i} = (X - 1)^m,$$

and hence $(\phi^\ell - 1)^m = 0$. By Lemma 6.5, this is equivalent to

$$[G, \underbrace{\phi^\ell, \dots, \phi^\ell}_{m\ \text{times}}] = 1. \tag{10}$$

Since G is abelian, we have $[G, G] = 1$, hence Eq. (10) implies that $\langle G, \phi^\ell \rangle$ has nilpotency class at most m. $\square$

By applying inductively the above lemma, we can prove a similar result for finitely generated nilpotent groups.

Lemma 7.3. *Let G be a finitely generated nilpotent group and let $\phi : G \to G$ be an automorphism of subexponential growth. Then there exists $\ell \in \mathbb{N} \setminus \{0\}$ with $\langle G, \phi^\ell \rangle$ nilpotent.*

Proof. We argue by induction on the nilpotency class c of G. If $c = 1$, that is $[G, G] = 1$, then G is abelian and Lemma 7.2 applies.

Suppose $c > 1$. By Lemma 7.2 and by the inductive hypothesis there exist $\ell_1, \ell_2 \in \mathbb{N} \setminus \{0\}$ such that both

$$\langle \gamma_c(G), \phi^{\ell_1} \rangle \quad \text{and} \quad \left\langle \frac{G}{\gamma_c(G)}, \phi^{\ell_2} \right\rangle$$

are nilpotent, say that the first has nilpotency class c_1 and the second nilpotency class c_2.

Write $\ell := \ell_1 \ell_2$. In view of Lemma 2.2 applied to $\langle G, \phi^\ell \rangle$ and to its normal subgroup $\gamma_c(G)$, we have $\gamma_{c_2+1}(\langle G, \phi^\ell \rangle) \leq \gamma_c(G)$. Moreover, since $[\gamma_c(G), G] = \gamma_{c+1}(G) = 1$, we get

$$\begin{aligned} \gamma_{c_1+c_2+1}(\langle G, \phi^\ell \rangle) &= [\gamma_{c_2+1}(\langle G, \phi^\ell \rangle), \underbrace{\langle G, \phi^\ell \rangle, \ldots, \langle G, \phi^\ell \rangle}_{c_1 \text{times}}] \\ &\leq [\gamma_c(G), \underbrace{\langle G, \phi^\ell \rangle, \ldots, \langle G, \phi^\ell \rangle}_{c_1 \text{times}}] \\ &\leq [\gamma_c(G), \underbrace{\phi^\ell, \ldots, \phi^\ell}_{c_1 \text{ times}}] \leq \gamma_{c_1+1}(\langle \gamma_c(G), \phi^\ell \rangle) = 1. \end{aligned}$$

Therefore $\langle G, \phi^\ell \rangle$ is nilpotent of nilpotency class at most $c_1 + c_2$. □

We are now in the position to prove the result announced at the beginning of this section.

Theorem 7.4. *If G is a locally virtually nilpotent group and $\phi : G \to G$ is an endomorphism, then ϕ has either exponential or polynomial growth.*

Proof. Assume that ϕ has subexponential growth. To conclude that ϕ has polynomial growth, we need to prove that, for every $F \in \mathcal{F}(G)$, the function $\gamma_{\phi,F}$ is polynomial. Therefore, fix $F \in \mathcal{F}(G)$ and set

$$V(\phi, F) := \langle F, \phi(F), \phi^2(F), \ldots, \phi^n(F), \ldots \rangle.$$

By Lemma 3.2 (a) $\phi \restriction_{V(\phi,F)}$ has subexponential growth. Clearly, $\gamma_{\phi,F}$ has polynomial growth if $\phi \restriction_{V(\phi,F)}$ has polynomial growth. In particular, we can assume without loss of generality that $G = V(\phi, F)$. By Lemma 3.3, G is finitely generated.

In view of Corollary 4.6 and Lemma 4.3, we may assume that ϕ is injective, hence ϕ is an automorphism of G by Proposition 4.1.

By Lemma 2.3 there exists a finite-index nilpotent normal ϕ-stable subgroup H of G, hence $\phi \restriction_H : H \to H$ is an automorphism; moreover H is finitely generated by Lemma 2.1. Since $\phi \restriction_H$ has subexponential growth by Lemma 3.2 (a), there exists $\ell \in \mathbb{N} \setminus \{0\}$ with $\langle H, \phi^\ell \rangle$ nilpotent by Lemma 7.3. Hence, ϕ has polynomial growth by Proposition 5.3 and Proposition 5.2. □

8. Growth of endomorphisms of locally virtually soluble groups

In this section we finally prove that, if G is a locally virtually soluble group, then every endomorphism of G has either polynomial or exponential growth.

The first step consists in proving that, if ϕ is an automorphism of a finitely generated soluble group G which is not polycyclic, then ϕ has exponential growth.

We follow Milnor [16] adapting his proof to our more general and slightly different situation.

We start with a rather technical lemma.

Lemma 8.1. *Let G be a finitely generated soluble group of derived length d and let $\phi : G \to G$ be an automorphism of subexponential growth. For each $\alpha \in G^{(d-1)}$ and for each $\beta \in G$, the set of conjugates*

$$\{(\beta\phi^{-1})^{-k}\alpha(\beta\phi^{-1})^k : k \in \mathbb{Z}\}$$

spans a finitely generated subgroup of $G^{(d-1)}$. (The computations are performed in $\langle G, \phi\rangle$.)

Proof. Set $A := G^{(d-1)}$ and observe that A is abelian. Let $\alpha \in A$ and $\beta \in G$. For each $m \in \mathbb{N} \setminus \{0\}$ and for each sequence $i_1, i_2, \dots, i_m$ with $i_j \in \{0, 1\}$, consider the expression

$$\alpha^{i_1}\beta\phi^{-1}\alpha^{i_2}\beta\phi^{-1}\cdots\alpha^{i_m}\beta\phi^{-1} \in \langle G, \phi\rangle. \tag{11}$$

We rewrite this expression in two different ways, each giving some useful insight. First, observe that $\phi^{-1}\gamma = \phi(\gamma)\phi^{-1}$ and, more generally, $\phi^{-t}\gamma = \phi^t(\gamma)\phi^{-t}$ for every $t \in \mathbb{Z}$. Thus Eq. (11) can also be written as

$$\alpha^{i_1}\beta\phi(\alpha^{i_2}\beta)\phi^2(\alpha^{i_3}\beta)\cdots\phi^{m-1}(\alpha^{i_m}\beta)\phi^{-m}. \tag{12}$$

Now set $\psi := \phi\beta^{-1}$, where we view $\psi : G \to G$ as the automorphism defined by $\psi(\gamma) := \beta\phi(\gamma)\beta^{-1}$, for every $\gamma \in G$.

Arguing as above, for each $\gamma \in G$ and $t \in \mathbb{Z}$, we have $\psi^{-t}\gamma = \psi^t(\gamma)\psi^{-t}$ for every $t \in \mathbb{Z}$. Thus Eq. (11) can also be written as

$$\alpha^{i_1}\psi(\alpha^{i_2})\cdots\psi^{m-1}(\alpha^{i_m})\psi^{-m}. \tag{13}$$

If for every $m \in \mathbb{N} \setminus \{0\}$ these 2^m expressions all represented distinct elements of $\langle G, \phi\rangle$, then Eq. (12) would give $|T_m(\phi, \{\beta, \alpha\beta\})| \geq 2^m$, contradicting the fact that ϕ does not have exponential growth.

Therefore there exist $m \in \mathbb{N}$, $i_1, \dots, i_m, j_1, \dots, j_m \in \{0, 1\}$ with $(i_1, \dots, i_m) \neq (j_1, \dots, j_m)$ such that the expressions in Eq. (11) corresponding to these strings give rise to the same element of $\langle G, \phi\rangle$. Observe that by choosing m as small as possible, we may assume that $i_1 \neq j_1$ and $i_m \neq j_m$, and that $m \geq 2$.

Now, by using Eq. (13), we obtain the equality

$$\alpha^{i_1}\psi(\alpha^{i_2})\psi^2(\alpha^{i_3})\cdots\psi^{m-1}(\alpha^{i_m}) = \alpha^{j_1}\psi(\alpha^{j_2})\psi^2(\alpha^{j_3})\cdots\psi^{m-1}(\alpha^{j_m}).$$

Therefore, using that A is a normal subgroup of $\langle G, \phi\rangle$ and that A is abelian, we get

$$\alpha^{i_1-j_1}\psi(\alpha^{i_2-j_2})\psi^2(\alpha^{i_2-j_2})\cdots\psi^{m-1}(\alpha^{i_m-j_m}) = e_G \tag{14}$$

where the exponents $i_k - j_k$ take the values 0 or 1 or -1, and are not all zero. In particular, by the minimality of m, $i_1 - j_1 \neq 0$ and $i_m - j_m \neq 0$.

From the equality in Eq. (14), we deduce first that $\psi^{m-1}(\alpha)$ is a word in $\alpha, \psi(\alpha), \dots, \psi^{m-2}(\alpha)$, and then that α is a word in $\psi(\alpha), \dots, \psi^{m-1}(\alpha)$; by applying ψ^{-1}, this in turn yields that $\psi^{-1}(\alpha)$ is a word in $\alpha, \psi(\alpha), \dots, \psi^{m-2}(\alpha)$.

From this it easily follows that $\psi^k(\alpha)$ is a word in $\alpha, \psi(\alpha), \dots, \psi^{m-2}(\alpha)$ for every $k \in \mathbb{Z}$. Recalling the definition of ψ we deduce that for every $k \in \mathbb{Z}$,

$$(\beta\phi^{-1})^{-k}\alpha(\beta\phi^{-1})^k \in \langle \alpha, (\beta\phi)^{-1}\alpha(\beta\phi^{-1})^{-1}, \dots, (\beta\phi^{-1})^{m-2}\alpha(\beta\phi^{-1})^{-(m-2)}\rangle. \qquad \square$$

We are now in the position to prove that an automorphism of a non-polycyclic finitely generated soluble group has exponential growth.

Theorem 8.2. *Let G be a finitely generated soluble group and let $\phi : G \to G$ be an automorphism of subexponential growth. Then G is polycyclic.*

Proof. Let $d \in \mathbb{N} \setminus \{0\}$ be the derived length of G. We argue by induction on d. The case $d = 1$ is trivial, since in this case G is a finitely generated abelian group and hence polycyclic.

Suppose that $d > 0$. To avoid some cumbersome notation, set $A := G^{(d-1)}$. Since G/A has derived length $d - 1$ and the automorphism induced by ϕ on G/A has subexponential growth in view of Lemma 3.2 (b), we conclude that G/A is polycyclic by the inductive hypothesis. In particular, since A is abelian, to deduce that G is polycyclic it suffices to show that A is finitely generated.

For $i \in \{0, \ldots, d - 1\}$, let

$$\bar{\phi}_i : G^{(i)}/G^{(i+1)} \to G^{(i)}/G^{(i+1)}$$

be the automorphism induced by ϕ on the section $G^{(i)}/G^{(i+1)}$. Observe that $G^{(i)}/G^{(i+1)}$ is abelian and $\bar{\phi}_i$ has subexponential growth by Lemma 3.2 (a) and (b). Therefore, by Lemma 7.2 there exists $\ell_i \in \mathbb{N} \setminus \{0\}$ such that ϕ^{ℓ_i} acts nilpotently on $G^{(i)}/G^{(i+1)}$. Set

$$\ell := \prod_{i=1}^{d} \ell_i.$$

By Lemma 6.3, ϕ^{ℓ} acts nilpotently on $G^{(i)}/G^{(i+1)}$ for every $i \in \{0, \ldots, d - 1\}$.

By Lemma 3.4, ϕ^{ℓ} has subexponential growth, so replacing ϕ by ϕ^{ℓ}, we may assume that ϕ acts nilpotently on each abelian section $G^{(i)}/G^{(i+1)}$.

As G/A is polycyclic and as ϕ acts nilpotently on each $G^{(i)}/G^{(i+1)}$, by Proposition 6.7 there exists a normal series

$$G = G_1 > G_2 > \cdots > G_{\kappa-1} > G_{\kappa} = A,$$

(1) refining the derived series of G/A;
(2) with G_i/G_{i+1} cyclic for each $i \in \{1, \ldots, \kappa - 1\}$;
(3) with $[G_i, \phi] \leq G_{i+1}$ for each $i \in \{1, \ldots, \kappa - 1\}$.

For each $i \in \{1, \ldots, \kappa - 1\}$, let $g_i \in G_i$ with $G_i = \langle g_i, G_{i+1}\rangle$, that is, g_iG_{i+1} is a generator for the cyclic group G_i/G_{i+1}. Since G/A is polycyclic, so is $\langle G, \phi\rangle/A$ and hence each element of $\langle G, \phi\rangle/A$ can be written as a product

$$g_1^{i_1} g_2^{i_2} \cdots g_{\kappa-1}^{i_{\kappa-1}} \phi^{i_\kappa} A \tag{15}$$

with $i_1, i_2, \ldots, i_{\kappa-1}, i_\kappa \in \mathbb{Z}$. We call $g_1, \ldots, g_{\kappa-1}, \phi$ a polycyclic presentation of $\langle G, \phi\rangle/A$. Furthermore, as $[G_i, \phi] \leq G_{i+1}$, we have

$$\phi(g_i) = g_i x_i, \tag{16}$$

for some $x_i \in G_{i+1}$.

We claim that each element x of the polycyclic group $\langle G, \phi\rangle/A$ can be also written as a product

$$x = (g_1\phi^{-1})^{i_1}(g_2\phi^{-1})^{i_2} \cdots (g_{\kappa-1}\phi^{-1})^{i_{\kappa-1}} \phi^{i_\kappa} A \tag{17}$$

with $i_1, i_2, \ldots, i_{\kappa-1}, i_\kappa \in \mathbb{Z}$. We prove this claim arguing by induction on κ, fixed the polycyclic presentation $g_1, \ldots, g_{\kappa-1}, \phi$.

Let $x \in \langle G, \phi\rangle/A$. By Eq. (15), we have $x = g_1^{j_1} g_2^{j_2} \cdots g_{\kappa-1}^{j_{\kappa-1}} \phi^{j_\kappa} A$, for some $j_1, \ldots, j_\kappa \in \mathbb{Z}$. Now, with a computation, we obtain

$$
\begin{aligned}
(g_1\phi^{-1})^{j_1} &= \underbrace{(g_1\phi^{-1})(g_1\phi^{-1})\cdots(g_1\phi^{-1})}_{j_1 \text{ times}} = g_1\phi(g_1)\phi^2(g_1)\cdots\phi^{j_1-1}(g_1)\phi^{-j_1}\\
&= g_1(g_1x_1)(g_1x_1\phi(x_1))\cdots(g_1x_1\phi(x_1)\cdots\phi^{j_1-2}(x_1))\phi^{-j_1},
\end{aligned}
$$

observe that in the second equality we used $\phi^{-\ell}g_1 = \phi^\ell(g_1)\phi^{-\ell}$ for each $\ell \in \{1, \ldots, j_1-1\}$, and in the third equality we used Eq. (16). Now, $x_1 \in G_2$ and hence $\phi^\ell(x_1) \in G_2$ for each ℓ. Moreover, $G_2 \trianglelefteq G_1 = G$. Therefore, by collecting all the g_1's on the left hand side, we obtain

$$(g_1\phi^{-1})^{j_1} = g_1^{j_1} y_1 \phi^{-j_1}, \tag{18}$$

for some $y_1 \in G_2$. Hence

$$g_1^{j_1} = (g_1\phi^{-1})^{j_1}\phi^{j_1}y_1^{-1},$$

and so

$$x = (g_1\phi^{-1})^{j_1}\phi^{j_1}y_1^{-1}g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}}\phi^{j_\kappa}A. \tag{19}$$

Observe that each element of $\langle G_2, \phi\rangle/A$ can be written as a product as in Eq. (15) with $i_1 = 0$. As $y_1^{-1}g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}} \in G_2$ and $\phi(G_2) = G_2$, we have

$$
\begin{aligned}
\phi^{j_1}y_1^{-1}g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}}\phi^{j_\kappa}A &= \phi^{-j_1}(y_1^{-1}g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}})\phi^{j_1+j_\kappa}A\\
&= g_2^{\ell_2}g_3^{\ell_3}\cdots g_{\kappa-1}^{\ell_{\kappa-1}}\phi^{\ell_\kappa}A,
\end{aligned}
$$

for some $\ell_2, \ldots, \ell_\kappa \in \mathbb{Z}$. By the inductive hypothesis applied to the group $\langle G_2, \phi\rangle/A$ with polycyclic presentation given by the elements $g_2, \ldots, g_{\kappa-1}, \phi$, there exist $i_2, \ldots, i_\kappa \in \mathbb{Z}$ with

$$g_2^{\ell_2}g_3^{\ell_3}\cdots g_{\kappa-1}^{\ell_{\kappa-1}}\phi^{\ell_\kappa}A = (g_2\phi^{-1})^{i_2}\cdots(g_{\kappa-1}\phi^{-1})^{i_{\kappa-1}}\phi^{i_\kappa}A. \tag{20}$$

Summing up, from Eqs. (19) and (20), we get

$$
\begin{aligned}
x &= g_1^{j_1}g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}}\phi^{j_\kappa}A = [(g_1\phi^{-1})^{j_1}\phi^{j_1}y_1^{-1}]g_2^{j_2}\cdots g_{\kappa-1}^{j_{\kappa-1}}\phi^{j_\kappa}A\\
&= (g_1\phi^{-1})^{j_1}(g_2\phi^{-1})^{i_2}\cdots(g_{\kappa-1}\phi^{-1})^{i_{\kappa-1}}\phi^{i_\kappa}A,
\end{aligned}
$$

and the claim in Eq. (17) is now proved.

In what follows we set $g_\kappa := e_G$. As $\langle G, \phi\rangle/A$ is polycyclic, it is finitely presented and hence, in view of Lemma 2.6, there exist $\alpha_1, \ldots, \alpha_\ell \in A$ such that every element of A can be expressed as a product of conjugates of the α_j in $\langle G, \phi\rangle$, that is,

$$A = \langle x\alpha_j x^{-1} : x \in \langle G, \phi\rangle, j \in \{1, \ldots, \ell\}\rangle. \tag{21}$$

Applying Lemma 8.1 with $\beta := g_\kappa$ and with each $\alpha \in \{\alpha_1, \ldots, \alpha_\ell\}$, we see that the set

$$\{(g_\kappa\phi^{-1})^{-k}\alpha_j(g_\kappa\phi^{-1})^k : k \in \mathbb{Z}, j \in \{1, \ldots, \ell\}\}$$

spans a finitely generated subgroup A_1 of A. Let $\alpha_{1,1}, \ldots, \alpha_{1,\ell_1}$ be a finite set of generators for A_1. Applying Lemma 8.1 with $g := g_{\kappa-1}$ and with each $\alpha \in \{\alpha_{1,1}, \ldots, \alpha_{1,\ell_1}\}$, we see that the set

$$\{(g_{\kappa-1}\phi^{-1})^{-k}\alpha_{1,j}(g_{\kappa-1}\phi^{-1})^k : k \in \mathbb{Z}, j \in \{1, \ldots, \ell_1\}\}$$

spans a finitely generated subgroup A_2 of A with $A_1 \le A_2$. Moreover, A_2 contains all the elements of the form

$$(g_{\kappa-1}\phi^{-1})^{-i_{\kappa-1}}(g_\kappa\phi^{-1})^{-i_\kappa}\alpha_j(g_\kappa\phi^{-1})^{i_\kappa}(g_{\kappa-1}\phi^{-1})^{i_{\kappa-1}},$$

with $i_{\kappa-1}, i_\kappa \in \mathbb{Z}$ and $j \in \{1,\dots,\ell\}$. Now arguing inductively we may construct a chain

$$A_1 \le A_2 \le A_3 \le \cdots \le A_{\kappa-1} \le A_\kappa$$

of finitely generated subgroups of A.

By construction, A_κ contains all elements of the form

$$(g_1\phi^{-1})^{-i_1}\cdots(g_{\kappa-1}\phi^{-1})^{-i_{\kappa-1}}(g_\kappa\phi^{-1})^{-i_\kappa}\alpha_j(g_\kappa\phi^{-1})^{i_\kappa}(g_{\kappa-1}\phi^{-1})^{i_{\kappa-1}}\cdots(g_1\phi^{-1})^{i_1},$$

with $i_1,\dots,i_{\kappa-1},i_\kappa \in \mathbb{Z}$ and $j \in \{1,\dots,\ell\}$. Since $i_1,\dots,i_\kappa$ are arbitrary integers, from Eq. (17), we deduce that A_κ contains all elements of the form

$$x\alpha_j x^{-1},$$

with $x \in \langle G, \phi\rangle$ and $j \in \{1,\dots,\ell\}$. Thus Eq. (21) yields $A = A_\kappa$ and A is finitely generated, hence G is polycyclic. □

Corollary 8.3. *Let G be a finitely generated virtually soluble group and let $\phi : G \to G$ be an automorphism of subexponential growth. Then there exists a finite-index polycyclic normal ϕ-stable subgroup H of G. In particular, G is virtually polycyclic.*

Proof. In view of Lemma 2.5 there exists a finite-index soluble normal ϕ-stable subgroup H of G. Moreover, H is finitely generated by Lemma 2.1 and $\phi\restriction_H$ has subexponential growth by Lemma 3.2 (a). Then H is polycyclic by Theorem 8.2, hence G is virtually polycyclic. □

The next lemma contains the fundamental part of the proof of Theorem 8.5.

Lemma 8.4. *Let G be a torsion-free polycyclic group and let $\phi : G \to G$ be an automorphism of subexponential growth. Then there exist $\ell \in \mathbb{N}\setminus\{0\}$ and a finite-index normal ϕ-stable subgroup N of G such that $\langle N, \phi^\ell\rangle$ is nilpotent.*

Proof. We proceed by induction on the Hirsch length of G. Our aim is to prove that G is virtually nilpotent and then Lemma 2.3 and Lemma 7.3 will give the thesis.

Consider the abelian quotient $G/[G,G]$. Then ϕ induces an automorphism on $G/[G,G]$ of subexponential growth by Lemma 3.2 (b). By Lemma 7.2, there exists $\ell \in \mathbb{N}\setminus\{0\}$ such that ϕ^ℓ acts nilpotently on $G/[G,G]$ and $\langle G/[G,G], \phi^\ell\rangle$ is nilpotent.

Let $t := |t(G/[G,G])|$ (so, t is the order of the torsion subgroup of the abelianization of G) and set

$$N := \langle [G,G], x^t : x \in G\rangle.$$

Then N is a normal ϕ-stable subgroup of G such that $[G,G] \le N \le G$ and $N/[G,G]$ is torsion-free. Since G is torsion-free, we have $N > [G,G]$ because $G/[G,G]$ cannot be a torsion group. Moreover, ϕ^ℓ acts nilpotently on $N/[G,G]$; so, as $N/[G,G]$ is a torsion-free abelian group, by Lemma 6.8 there exists a ϕ^ℓ-stable subgroup M of G with $[G,G] \le M < N$ and N/M infinite cyclic. As M contains $[G,G]$, we see that M is a normal subgroup of G.

By construction, the Hirsch length of G is strictly larger (by one) than the Hirsch length of M. Therefore, by induction, there exists a non-zero multiple ℓ' of ℓ and there exists a finite-index ϕ^ℓ-stable normal subgroup K of M such that $\langle K, \phi^{\ell'}\rangle$ is nilpotent.

Take

$$C := \bigcap_{g\in G} K^g.$$

Clearly, C is a ϕ^ℓ-stable normal subgroup of G and $\langle C, \phi^{\ell'}\rangle$ is nilpotent. Moreover, C has finite-index in M; this is because there is a natural injection

$$\frac{M}{C} = \frac{M}{\bigcap_{g\in G} K^g} \to \prod_{g\in G} \frac{M}{K^g},$$

where $\prod_{g\in G} M/K^g$ has exponent at most $|M : K|$ and so M/C must be finite since M is polycyclic.

Consider the following sequence of $\phi^{\ell'}$-stable normal subgroups of G

$$C \le M < N \le G,$$

and observe that G/C is polycyclic and it is actually an extension of the finite group M/C, by the infinite cyclic group N/M, by the finite group G/N. By Lemma 2.7 there exists a finite-index $\phi^{\ell'}$-stable normal subgroup L of G with L/C torsion-free; in our situation this means that L/C is infinite cyclic. See Figure 1.

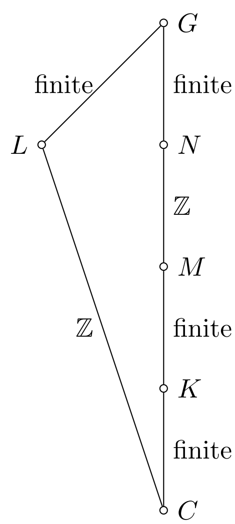

Figure 1. Figure for the proof of Lemma 8.4

Since our aim is to prove that G is virtually nilpotent, to simplify our notation we replace G by L, M by C and ϕ by $\phi^{\ell'}$. In particular, now G is a torsion-free polycyclic group that contains a normal ϕ-stable subgroup M with G/M infinite cyclic. Moreover, ϕ acts nilpotently on M and $\langle M, \phi\rangle$ is nilpotent, and hence M is nilpotent.

Let $x \in G$ with $G = \langle x, M\rangle$. As ϕ normalizes $G/M \cong \mathbb{Z}$, we see that $\phi(xM) = x^\varepsilon M$, where $\varepsilon \in \{-1, 1\}$. In particular, replacing ϕ by ϕ^2 we may now assume that $\varepsilon = 1$. In particular, $\phi(x) = xm$, for some $m \in M$. Given $k \in \mathbb{N} \setminus \{0\}$, set $m_k := m\phi(m)\cdots\phi^{k-1}(m)$. Arguing inductively, it is easy to show that

$$\phi^k(x) = xm_k \quad \text{and} \quad m_k \in M, \tag{22}$$

for every $k \in \mathbb{N} \setminus \{0\}$.

Let A be the last term of the lower central series of M, thus A is abelian and central in M. Set

$$C := C_A(\phi) = \{a \in A : \phi(a) = a\} \leq A \leq M.$$

As ϕ acts nilpotently on A and A is torsion-free, we see that C is infinite and torsion-free. In fact, there exists a minimum $k \in \mathbb{N} \setminus \{0\}$ such that $[A,_k \phi] = 1$, so $[A,_{k-1} \phi] \neq 1$ (by Lemma 6.5 we have $[A,_n \phi] = (\phi - id_A)^n(A)$ for every $n \in \mathbb{N} \setminus \{0\}$, hence $C \geq [A,_{k-1} \phi]$). Moreover, C is a normal subgroup of G; in fact, $C \trianglelefteq M$ and

$$\begin{aligned} C^x &= (C_A(\phi))^x = C_{A^x}(x^{-1}\phi x) = C_A(\phi \cdot (\phi^{-1}x^{-1}\phi)x) \\ &= C_A(\phi \cdot \phi(x^{-1})x) = C_A(\phi \cdot (xm)^{-1}x) = C_A(\phi m) = C_A(\phi) = C, \end{aligned}$$

where the equality $C_A(\phi m) = C_A(\phi)$ follows because $m \in M$ and A is central in M.

Since the Hirsch length of G/C is strictly smaller than the Hirsch length of G, by the inductive hypothesis there exist $\ell \in \mathbb{N} \setminus \{0\}$ and a finite-index normal ϕ-stable subgroup K of G with $C \leq K$ and $\langle K/C, \phi^\ell \rangle$ nilpotent.

As above, since our aim is to prove that G is virtually nilpotent, we assume that $G = K$ and $\ell = 1$, replacing also M with $M \cap K$. Therefore, G is a torsion-free polycyclic group and we have a chain

$$C \leq M \leq G$$

with

(i) $G/M \cong \mathbb{Z}$,
(ii) $G = \langle x, M \rangle$,
(iii) $\langle G/C, \phi \rangle$ nilpotent (and hence G/C nilpotent),
(iv) C central in $\langle M, \phi \rangle$,
(v) $\langle M, \phi \rangle$ nilpotent (and hence M nilpotent).

Let $\iota_x : C \to C$ be the automorphism induced by x by conjugation on C (i.e., ι_x is the restriction to C of the inner automorphism of G relative to x).

Let $F \in \mathcal{F}(C)$ and let $m \in \mathbb{N} \setminus \{0\}$. We prove that

$$\gamma_{\iota_{x^{-1}}, F}(m+1) = \gamma_{\phi, Fx}(m+1). \tag{23}$$

Consider $\alpha_0, \ldots, \alpha_m \in F$ and

$$y := \alpha_0 x \phi(\alpha_1 x) \phi^2(\alpha_2 x) \cdots \phi^m(\alpha_m x) \in T_{m+1}(\phi, Fx).$$

Since ϕ centralizes C (i.e., $\phi(c) = c$ for every $c \in C$), the above product becomes

$$y = \alpha_0 x \alpha_1 \phi(x) \alpha_2 \phi(x^2) \cdots \alpha_m \phi^m(x).$$

For every $k \in \mathbb{N} \setminus \{0\}$ and $\alpha \in C$, from Eq. (22) we have

$$\phi^k(x)\alpha = x m_k \alpha = x \alpha m_k = x \alpha x^{-1} \cdot x m_k = \iota_{x^{-1}}(\alpha) x m_k = \iota_{x^{-1}}(\alpha) \phi^k(x)$$

and hence

$$\phi^k(x)\alpha = \iota_{x^{-1}}(\alpha)\phi^k(x). \tag{24}$$

Using Eq. (24), we may rewrite our product y by pushing all the x's on the right, obtaining

$$\begin{aligned} y &= \alpha_0 \iota_{x^{-1}}(\alpha_1) \iota^2_{x^{-1}}(\alpha_2) \cdots \iota^m_{x^{-1}}(\alpha_m) x \phi(x) \phi^2(x) \cdots \phi^m(x) \\ &\in T_{m+1}(\iota_{x^{-1}}, F) x \phi(x) \phi^2(x) \cdots \phi^m(x). \end{aligned}$$

This proves that
$$T_{m+1}(\phi, Fx) = T_{m+1}(\iota_{x^{-1}}, F)x\phi(x)\phi^2(x)\cdots\phi^m(x).$$
Now,
$$|T_{m+1}(\phi, Fx)| = |T_{m+1}(\iota_{x^{-1}}, F)x\phi(x)\phi^2(x)\cdots\phi^m(x)| = |T_{m+1}(\iota_{x^{-1}}, F)|,$$
and this proves the equality in Eq. (23).

Since ϕ has subexponential growth, from Eq. (23) we deduce that $\gamma_{\iota_{x^{-1}},F}$ is subexponential, and by the arbitrariness of $F \in \mathcal{F}(C)$ we conclude that $\iota_{x^{-1}}$ (and hence ι_x) has subexponential growth.

From Lemma 7.2, applied with $G := C$ and $\phi := \iota_x$, there exists $\ell \in \mathbb{N} \setminus \{0\}$ such that $\iota_x^\ell = \iota_{x^\ell}$ acts nilpotently on C. As usual, replacing ϕ^ℓ by ϕ, we may assume that $\ell = 1$. Let $c \in \mathbb{N} \setminus \{0\}$ with $[C,_c \iota_x] = 1$.

From (iii), G/C is nilpotent and hence there exists $d \in \mathbb{N}$ with $\gamma_{d+1}(G/C) = 1$. Then $\gamma_{d+1}(G) \leq C$ by Lemma 2.2. Since $G = \langle x, M\rangle$ and C is central in M (see (ii) and (iv)), we obtain
$$\begin{aligned}\gamma_{d+c+1}(G) = [\gamma_{d+1}(G), \underbrace{G, \ldots, G}_{c \text{ times}}] &\leq [C, \underbrace{G, \ldots, G}_{c \text{ times}}] \\ &= [C, \underbrace{\langle x, M\rangle, \ldots, \langle x, M\rangle}_{c \text{ times}}] = [C,_c \iota_x] = 1.\end{aligned}$$
This implies that G is nilpotent. □

We are now in the position to prove our main theorem.

Theorem 8.5. *If G is a locally virtually soluble group and $\phi : G \to G$ is an endomorphism, then ϕ has either exponential or polynomial growth.*

Proof. Assume that ϕ has subexponential growth. To conclude that ϕ has polynomial growth, we need to prove that, for every $F \in \mathcal{F}(G)$, the function $\gamma_{\phi,F}$ has polynomial growth. Therefore, fix $F \in \mathcal{F}(G)$ and set $V(\phi, F) := \langle F, \phi(F), \phi^2(F), \ldots\rangle$. By Lemma 3.2 (a) $\phi \restriction_{V(\phi,F)}$ has subexponential growth. Clearly, $\gamma_{\phi,F}$ has polynomial growth if $\phi \restriction_{V(\phi,F)}$ has polynomial growth. In particular, we may assume without loss of generality that $G = V(\phi, F)$. By Lemma 3.3, G is finitely generated.

Let $K := \mathrm{Ker}_\infty(\phi)$ and let $\bar\phi : G/K \to G/K$ be the endomorphism induced by ϕ on G/K. Observe that $\bar\phi$ is injective. By Lemma 3.2 (b), $\bar\phi$ has subexponential growth, and hence Proposition 4.1 yields that $\bar\phi$ is an automorphism. Then $\bar\phi$ is an automorphism of a finitely generated virtually soluble group of subexponential growth.

Corollary 8.3 yields that G/K has a finite-index polycyclic $\bar\phi$-stable normal subgroup H/K. In particular G/K is virtually polycyclic, hence finitely presented. By Lemma 4.7 and Lemma 4.3, we conclude that $\bar\phi$ and ϕ have the same growth type.

Replacing G by G/K if necessary, we may assume that G is virtually polycyclic, ϕ is an automorphism of subexponential growth and H is a finite-index polycyclic normal ϕ-stable subgroup of G. By Lemma 2.7, we may assume that H is torsion-free.

By Lemma 8.4, there exist $\ell \in \mathbb{N} \setminus \{0\}$ and a finite-index normal ϕ^ℓ-stable subgroup N of H such that $\langle N, \phi^\ell\rangle$ is nilpotent. Since $|\langle G, \phi\rangle : \langle N, \phi^\ell\rangle| = |G : N|\ell$ is finite, the group $\langle G, \phi\rangle$ is virtually nilpotent. Hence ϕ has polynomial growth by Proposition 5.2. □

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