Algebraic Actions of the Discrete Heisenberg Group

Based on joint work with Doug Lind and Evgeny Verbitskiy

Klaus Schmidt

Vienna

DAGT, Udine, July 2018

Toral automorphisms

Let α be the automorphism of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ given by a matrix $A \in \mathrm{GL}(n, \mathbb{Z})$. Then α is *ergodic* (with respect to Lebesgue measure) if and only if no eigenvalue of A is a root of unity, and *expansive* (or *hyperbolic*) if and only if no eigenvalue of A has absolute value 1.

A point $x \in \mathbb{T}^n$ is homoclinic under α if $\lim_{|n|\to\infty} \alpha^n x = 0$. Hyperbolic toral automorphisms have nonzero homoclinic points, Markov partitions (Sinai, '68; Bowen, '70), hence both *periodic* and *homoclinic specification*, and the probability measures defined by partial or periodic orbits are dense in the set of all invariant probability measures.

Nonexpansive ergodic toral automorphisms (assumed to be *irreducible* for simplicity) have no nonzero homoclinic points and no Markov partitions.

However, they satisfy *weak specification* (Lind, '79), and the periodic orbit measures are again dense in the set of all invariant probability measures (Marcus, '80).

Irreducible toral automorphisms

Every irreducible matrix $A \in \mathrm{GL}(n, \mathbb{Z})$ is conjugate over $\mathbb{Q}$ to a companion matrix of the form

$$A_f = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ f_0 & f_1 & f_2 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix},$$

where $f = t^n - f_{n-1}t^{n-1} - \cdots - f_0$ is the (irreducible) characteristic polynomial of A.

If $(\sigma x)_k = x_{k+1}$ is the shift on $\mathbb{T}^{\mathbb{Z}}$, then

$$X_f = \ker f(\sigma) = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : x_{k+n} - f_{n-1}x_{k+n-1} - \cdots - f_0 x_k = 0 \ \text{for all } k\} \quad (\star)$$

is a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$, and the restriction α_f of σ to X_f is conjugate to the toral automorphism defined by A_f.

In other words, every irreducible toral automorphism is *finitely equivalent* to a shift α_f on $X_f \subset \mathbb{T}^{\mathbb{Z}}$ of the form $(\star)$, where f is an irreducible monic polynomial with integer coefficients and $|f_0| = 1$.

Irreducible automorphisms of compact abelian groups

Equation $(\star)$ also makes sense if f is not monic: if $f = f_n t^n - \cdots - f_0$ is an integer polynomial with $f_0 f_n \neq 0$, then

$$X_f = \ker f(\sigma) = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : f_n x_{k+n} - f_{n-1} x_{k+n-1} - \cdots - f_0 x_k = 0 \text{ for all } k\}$$

is again a closed, shift-invariant subgroup of $\mathbb{T}^{\mathbb{Z}}$. If f is irreducible, the restriction α_f of σ to X_f is *irreducible* in the sense that every closed, invariant subgroup $Y \subsetneq X_f$ is finite.

Up to finite equivalence, every irreducible automorphism of a compact abelian group can be represented in this way.

Basic dynamical properties of α_f, like *expansiveness* or *entropy*, are again determined by the roots of f. In particular, α_f is expansive and has nonzero homoclinic points if and only if f has no roots of absolute value 1.

The entropy of α_f is given by

$$h(\alpha_f) = \log |f_n| + \sum_{\{\lambda : f(\lambda)=0 \text{ and } |\lambda|>0\}} \log |\lambda| = \int_{\mathbb{S}} \log |f(s)| \, ds,$$

where $\mathbb{S} = \{s \in \mathbb{C} : |s| = 1\}$.

Two elementary examples

(1) Let $f = 2$. Then

$$X_f = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : 2x_k = 0 \text{ for all } k\}.$$

A homoclinic point $v \in X_f$ is given by

$$v_n = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) Let $f = 2t - 3$. Then

$$X_f = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : 2x_{k+1} = 3x_k \text{ for all } k\},$$

α_f is 'multiplication by 3/2' on $\mathbb{T}$, and $h(\alpha_f) = \log 3$. The point $v \in X_f$ with

$$v_n = \begin{cases} 2^n/3^n & \text{if } n < 0, \\ 0 & \text{otherwise,} \end{cases}$$

is homoclinic.

Two nonexpansive examples

(1) Let $A = \left(\begin{smallmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{smallmatrix}\right) \in \mathrm{GL}(4, \mathbb{Z})$. The toral automorphism α defined by A is conjugate to α_f with $f = t^4 - t^3 - t^2 - t + 1$. Since f is irreducible, noncyclotomic, and has two roots of absolute value 1, α_f is ergodic and nonexpansive, and has no nonzero homoclinic points.

(2) Let $f = 5t^2 - 6t + 5$. Then

$$X_f = \{(x_k) \in \mathbb{T}^{\mathbb{Z}} : 5x_{k+2} - 6x_{k+1} + 5x_k = 0 \text{ for all } k\}.$$

The roots of f has absolute value 1, α_f is nonexpansive, $h(\alpha_f) = \log 5$, and there are no nonzero homoclinic points.

What are homoclinic points good for?

If α_f is expansive, all homoclinic points of α_f decay exponentially. For every homoclinic point $x \in X_f$ we can thus define a shift-equivariant map $\xi_x \colon \ell^\infty(\mathbb{Z}, \mathbb{Z}) \longrightarrow X_f$ by setting

$$\xi_x(w) = \sum_{n \in \mathbb{Z}} w_n \alpha_f^{-n} x, \quad w = (w_n) \in \ell^\infty(\mathbb{Z}, \mathbb{Z}).$$

The map ξ_x is weak*-continuous on closed, bounded subsets of $\ell^\infty(\mathbb{Z}, \mathbb{Z})$. For sufficiently large $N \geq 1$, $\xi_x(\{0, \dots, N\}^{\mathbb{Z}}) = X_f$, so that α_f is a continuous factor of a full shift.

Some homoclinic points are particularly useful: a homoclinic point $v \in X_f$ is *fundamental* if every homoclinic point of α_f lies in the subgroup of X_f generated by the orbit $\{\alpha_f^n v : n \in \mathbb{Z}\}$ of v under α_f.
If the homoclinic point $v \in X_f$ is fundamental, one can choose a closed, bounded, shift-invariant subset $W \subset \ell^\infty(\mathbb{Z}, \mathbb{Z})$ which is a sofic shift, such that $\xi_v(W) = X_f$ and $\xi_v|_W \colon W \longrightarrow X_f$ is almost one-to-one. In other words, fundamental homoclinic points yield *sofic partitions* of (X_f, α_f) (Vershik '92, Einsiedler-Schmidt '97, Kenyon-Vershik '98, ...).

A formula for the fundamental homoclinic point of α_f

We write Ω_f for the set of roots of f and consider the partial fraction decomposition

$$\frac{1}{f(t)} = \frac{1}{f_m} \sum_{\omega \in \Omega_f} \frac{b_\omega}{t - \omega}$$

of $1/f$ with $b_\omega \in \mathbb{C}$ for every $\omega \in \Omega_f$. Define $w^\Delta \in \ell^\infty(\mathbb{Z}, \mathbb{R})$ by

$$w_n^\Delta = \begin{cases} \frac{1}{f_m} \cdot \sum_{\omega \in \Omega_f^-} \quad b_\omega \omega^{n-1} & \text{if } n \geq 1, \\ \frac{1}{f_m} \cdot \sum_{\omega \in \Omega_f^+} -b_\omega \omega^{n-1} & \text{if } n \leq 0, \end{cases}$$

where Ω_f^+ and Ω_f^- denote the set of large, resp. small, roots of f.

Then

$$x^\Delta = w^\Delta \pmod 1$$

lies in X_f and is 'the' fundamental homoclinic point of α_f.

For example, if $f = t^2 - t - 1$, then $x^\Delta \in X_f$ is given by

$$x_n^\Delta = \begin{cases} -\frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2}\right)^n \pmod 1 & \text{if } n \geq 1, \\ -\frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2}\right)^n \pmod 1 & \text{if } n \leq 0. \end{cases}$$

Periodic points of α_f

For every $k \geq 1$, the set of points $\mathsf{Fix}_k(\alpha_f) = \{x \in \mathbb{T}^n : \alpha^k x = x\}$ with period k under α_f satisfies that

$$|\mathsf{Fix}_k(\alpha_f)| = \prod_{\{\omega \in \mathbb{C} : \omega^k = 1\}} |f(\omega)|.$$

In particular, if f is ergodic, $\mathsf{Fix}_k(\alpha_f)$ is finite for every $k \geq 1$.

Furthermore,

$$\begin{aligned}\lim_{k \to \infty} \frac{1}{k} \log \mathsf{Fix}_k(\alpha_f) &= \lim_{k \to \infty} \frac{1}{k} \sum_{\{\omega \in \mathbb{C} : \omega^k = 1\}} \log |f(\omega)| \\ &= \int_{\mathbb{S}} \log |f(s)| \, ds = h(\alpha_f).\end{aligned}$$

If α_f is expansive, this is obvious. If α_f is nonexpansive, this requires a diophantine result due to Gelfond '32.

What if α_f is nonexpansive?

If f has roots of absolute value 1, there are no nonzero homoclinic points, so that the approach just described is unavailable. However, by using either a geometric approach (Lind, Marcus, around '80) or an algebraic approach (Lindenstrauss-S '05, S '06 and '16) one can, to a limited extent, rescue some of the 'expansive' results for nonexpansive automorphisms.

What is interesting is that the *number* of unitary roots of f (i.e., the size of the *unitary variety* $\mathrm{U}(f)$ of f) seems to play no role in these partial results.

For the corresponding $\mathbb{Z}^d$-actions by automorphisms of compact abelian groups, which are again determined by an integer polynomial f in $d \geq 2$ variables, things are quite different. Here the size of the unitary variety

$$\mathrm{U}(f) = \{\mathbf{z} = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_i| = 1 \text{ for } i = 1, \dots, d \text{ and } f(\mathbf{z}) = 0\}$$

of f *will* play an important role.

Algebraic $\mathbb{Z}^d$-actions

Let $d \geq 1$, and let $\alpha\colon \mathbf{n} \to \alpha^{\mathbf{n}}$ be a $\mathbb{Z}^d$-action by automorphisms of a compact abelian group X.

If $h(\alpha) > 0$, then X contains a closed, α-invariant subgroup Y such that (Y, α) is algebraically conjugate to the shift-action α_f of $\mathbb{Z}^d$ on a closed, shift-invariant subgroup $X_f \subset \mathbb{T}^{\mathbb{Z}^d}$ of the form

$$X_f = \ker f(\sigma) = \Big\{(x_{\mathbf{n}}) \in \mathbb{T}^{\mathbb{Z}^d} : \textstyle\sum_{\mathbf{m}\in\mathbb{Z}^d} f_{\mathbf{m}} x_{\mathbf{m}+\mathbf{n}} = 0 \text{ for every } \mathbf{n} \in \mathbb{Z}^d\Big\},$$

where $f = \sum_{\mathbf{m}\in\mathbb{Z}^d} f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}}$ is a polynomial in d variables $u_1, \dots, u_d$ with integer coefficients (here $\mathbf{u}^{\mathbf{m}} = u_1^{m_1} \cdots u_d^{m_d}$).

We write R_d for the ring of all (Laurent) polynomials in d variables with integer coefficients and consider R_d as a subring of the convolution algebra $\ell^1(\mathbb{Z}^d, \mathbb{R})$.

Properties of (X_f, α_f)

Let $d > 1$ and $0 \neq f \in R_d$.

- (X_f, α_f) is ergodic.
- (X_f, α_f) is expansive if and only if $\mathrm{U}(f) = \varnothing$ (S '90).
- (X_f, α_f) is expansive if and only if the polynomial f is invertible in $\ell^1(\mathbb{Z}^d, \mathbb{R})$ (Wiener '32, Deninger-S '07). In this case $1/f^* \pmod 1$ lies in X_f and is a fundamental homoclinic point of α_f (here $f^* = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} \mathbf{u}^{-\mathbf{m}}$).

Example: A (Laurent) polynomial $f = \sum_{\mathbf{m} \in \mathbb{Z}^d} f_{\mathbf{m}} \mathbf{u}^{\mathbf{m}}$ is *lopsided* if there exists an $\mathbf{m} \in \mathbb{Z}^d$ such that $|f_{\mathbf{m}}| > \sum_{\mathbf{n} \in \mathbb{Z}^d \smallsetminus \{\mathbf{m}\}} |f_{\mathbf{n}}|$. If f is lopsided, then α_f is expansive. In fact, α_f is expansive if and only if the principal ideal $\langle f \rangle = R_d f \subset R_d$ contains a lopsided polynomial (observation by Hanfeng Li).

Entropy of (X_f, α_f)

Suppose that $d > 1$ and $f \in R_d$.

- If f is nonzero, then
$$h(\alpha_f) = \int_{\mathbb{S}^d} \log |f(\mathbf{s})| \, d\mathbf{s}$$
(Lind-S-Ward '90).
- If $f = 0$, then $h(\alpha_f) = \infty$.
- If $f \neq 0$, α_f has positive entropy if and only if f is not a product of generalized cyclotomic polynomial (Lind-S-Ward '90).
- If $f \neq 0$, $h(\alpha_f)$ coincides with the upper logarithmic growth rate of the number of connected components of points with finite orbits:
$$h(\alpha_f) = \limsup_{\langle\Delta\rangle\to\infty} \tfrac{1}{|\mathbb{Z}^d/\Delta|} \log |\mathrm{Fix}_\Delta(\alpha_f)/\mathrm{Fix}^\circ_\Delta(\alpha_f)|, \qquad (\bullet)$$
where $\mathrm{Fix}^\circ_\Delta(\alpha_f)$ is the connected component of the identity in the group $\mathrm{Fix}_\Delta(\alpha_f)$ of Δ-periodic points in X_f, and where the lim sup is taken over all finite-index subgroups $\Delta \subset \mathbb{Z}^d$ as $\langle\Delta\rangle = \min\{\|\mathbf{n}\| : \mathbf{n} \in \Delta \smallsetminus \{\mathbf{0}\}\} \to \infty$ (S '95).

Homoclinic points of α_f

Suppose that $d > 1$ and $f \in R_d$ is nonzero and irreducible. By definition, every homoclinic point $x = (x_{\mathbf{n}}) \in X_f \subset \mathbb{T}^{\mathbb{Z}^d}$ satisfies that

$$\lim_{\mathbf{n}\to\infty} |x_{\mathbf{n}}| = 0.$$

We write $\Delta_{\alpha_f}(X_f)$ for the group of homoclinic points in X_f. A homoclinic point $x \in \Delta_{\alpha_f}(X_f)$ is *summable* if $\sum_{\mathbf{n}\in\mathbb{Z}^d} |x_{\mathbf{n}}| < \infty$. The group of summable homoclinic points of α_f is denoted by $\Delta^1_{\alpha_f}(X_f)$.

(a) If α_f is expansive, then $\Delta^1_{\alpha_f}(X_f)$ is countable and dense in X_f, and $\Delta_{\alpha_f}(X_f) = \Delta^1_{\alpha_f}(X_f)$ (Lind-S '99).

(b) If α_f is nonexpansive, $\Delta^1_{\alpha_f}(X_f)$ is either trivial or countable and dense in X_f (Lind-S '99).

(c) If $\mathrm{U}(f)$ is not contained in a finite union of (d-1)-dimensional subgroups of $\mathbb{S}^d$, then $\Delta_{\alpha_f}(X_f)$ is uncountable (Linnell-Puls '01).

(d) If $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$ then $0 < h(\alpha_f) < \infty$.

(e) If $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$, then

$$h(\alpha_f) = \lim_{\langle\Delta\rangle\to\infty} \frac{1}{|\mathbb{Z}^d/\Delta|} \log |\mathrm{Fix}_\Delta(\alpha_f)/\mathrm{Fix}^\circ_\Delta(\alpha_f)|. \qquad (\bullet\bullet)$$

Existence of summable homoclinic points

Theorem (Lind-S-Verbitskiy '13) Suppose that $d > 1$ and $f \in R_d$ is nonzero and irreducible. Then $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$ if and only if $\dim \mathrm{U}(f) \leq d - 2$.

Examples

- If f is asymmetric (i.e., if $\gcd(f, f^*) = 1$), then $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$ (Lind-S-Verbitskiy '13).
- *The harmonic system*: Let $f = 4 - u - v - u^{-1} - v^{-1} \in R_2$. Then $\mathrm{U}(f) = \{(1, 1)\}$, and $0 = \dim \mathrm{U}(f) = d - 2$. Hence $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$ (S-Verbitskiy '09).
- Let $f = 1 + u + v + w \in R_3$. Then $1 = \dim \mathrm{U}(f) = 3 - 2$, so $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$.
- Let $f = 3 - u - v - u^{-1} - v^{-1} \in R_2$. then $1 = \dim \mathrm{U}(f) > 2 - 2$, so $\Delta^1_{\alpha_f}(X_f) = \{0\}$, but $\Delta_{\alpha_f}(X_f)$ is uncountable. For this example, the specification properties of α_f are not understood.

Principal Algebraic Actions of Discrete Groups

Let Γ be a countable discrete group. We consider the shift actions λ and ρ of Γ on $\mathbb{T}^\Gamma$, where

$$(\lambda^\gamma x)_{\gamma'} = x_{\gamma^{-1}\gamma'}, \qquad (\rho^\gamma x)_{\gamma'} = x_{\gamma'\gamma}$$

for every $\gamma \in \Gamma$ and $x = (x_{\gamma'}) \in \mathbb{T}^\Gamma$.

To get so-called *principal* actions, consider the integer group ring $\mathbb{Z}\Gamma$ of Γ. Write $f \in \mathbb{Z}\Gamma$ as a formal sum $f = \sum_{\gamma\in\Gamma} f_\gamma \cdot \gamma$ and set

$$\lambda^f = \sum_{\gamma\in\Gamma} f_\gamma \lambda^\gamma, \qquad \rho^f = \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma.$$

Then

$$X_f := \ker \rho^f = \Big\{x \in \mathbb{T}^\Gamma : \sum_{\gamma\in\Gamma} f_\gamma \rho^\gamma x = 0\Big\}$$

is a closed, λ-invariant subgroup of $\mathbb{T}^\Gamma$ (it is the kernel of right convolution by $f^* = \sum_{\gamma\in\Gamma} f_\gamma \cdot \gamma^{-1}$). We denote by α_f the restriction of the left shift-action λ to X_f and call (X_f, α_f) the principal Γ-action defined by f.

There are very few general results about principal algebraic actions of discrete groups.

Expansiveness: Let Γ be a discrete group and $f \in \mathbb{Z}\Gamma$. Then α_f is expansive if and only if f is invertible in $\ell^1(\Gamma)$ (Deninger-S, '07).

Entropy: if Γ is sofic, $f \in \mathbb{Z}\Gamma$, and ρ^f is injective on $\ell^2(\Gamma)$, then $h(\alpha_f) = \log \det(f)$, the Fuglede-Kadison determinant of f, acting by convolution on $\ell^2(\Gamma)$ (Sinai '59, Lind-S-Ward '90; Deninger '06, Deninger-S '07, Li '12, Li-Thom '14; Bowen, Hayes, Kerr, Li, ...).

If ρ^f in noninjective on $\ell^2(\Gamma)$, $h(\alpha_f) = \infty$ (Chung-Li '15).

If α_f is expansive or, more generally, if α_f has a nontrivial summable homoclinic point, then $h(\alpha_f) > 0$.

Let $\mathbb{H} = \left\{ \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix} \right) : a, b, c \in \mathbb{Z} \right\}$ be the discrete Heisenberg group, with generators

$$x = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix} \right), \quad y = \left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right), \quad \text{and} \quad z = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right), \quad \text{where} \quad yx = xyz.$$

In this notation, every $f \in \mathbb{Z}\mathbb{H}$ can be written as an integer polynomial

$$f = \sum\nolimits_{k,l,m} f_{k,l,m} x^k y^l z^m$$

in the noncommuting variables x, y, z.

Theorem (Einsiedler-Rindler, '01). Fix $f \in \mathbb{Z}\mathbb{H}$. The action α_f is nonexpansive if and only if there exist an irreducible unitary representation π of $\mathbb{H}$ on a Hilbert space $\mathcal{H}$ and a unit vector $v \in \mathcal{H}$ such that $\pi(f)v = 0$.

Problem: $\mathbb{H}$ is not a Type I group. So the set of irreducible unitary representations of $\mathbb{H}$ is complicated.

An application of Allan's local principle

Theorem (Göll-S-Verbitskiy, '16). Let $f \in \mathbb{Z}\mathbb{H}$. Then α_f is nonexpansive if and only if $\pi(f)$ is invertible for every irreducible representation π of $\mathbb{H}$ which is induced from a one-dimensional representation of a subgroup of $\mathbb{H}$ (such representations are called monomial).

The following examples are taken from Einsiedler-Rindler '01 and Göll-S-Verbitskiy, '14.

- $f = 3 + x + y + z$ is invertible in $\ell^1(\mathbb{H})$, but not in $\ell^1(\mathbb{Z}^3)$.
- $f = 3 + x + y - z$ is noninvertible in $\ell^1(\mathbb{H})$ as well as in $\ell^1(\mathbb{Z}^3)$.
- $f = 3 \pm 2x \pm y + z$ is invertible in $\ell^1(\mathbb{H})$, but not in $\ell^1(\mathbb{Z}^3)$.

Problem: Find an effective criterion for expansiveness of principal actions of $\mathbb{H}$.

Homoclinic and periodic points of nonexpansive actions

For $f \in \mathbb{Z}[\mathbb{Z}^d]$ we saw earlier that α_f may be nonexpansive, but $\Delta^1_{\alpha_f}(X_f)$ may nevertheless be nontrivial. The same can happen for $f \in \mathbb{Z}\mathbb{H}$.

Examples (Göll-S-Verbitskiy, '16)

- $f = 2 - x - y$.
- $f = 4 - x - x^{-1} - y - y^{-1}$.

Currently it is not known which nonexpansive principal actions of $\mathbb{H}$ have summable homoclinic points.

If α_f is expansive, or if $\Delta^1_{\alpha_f}(X_f) \neq \{0\}$, then the entropy of α_f coincides with the logarithmic growth rate of the number of its periodic points:

$$h(\alpha_f) = \lim_{\langle\Delta\rangle\to\infty} \frac{1}{|\mathbb{H}/\Delta|} \log |\mathrm{Fix}_\Delta(\alpha_f)/\mathrm{Fix}^\circ_\Delta(\alpha_f)|.$$

If α_f is nonexpansive, then it is easy to see that

$$h(\alpha_f) \geq \limsup_{\langle\Delta\rangle\to\infty} \frac{1}{|\mathbb{H}/\Delta|} \log |\mathrm{Fix}_\Delta(\alpha_f)/\mathrm{Fix}^\circ_\Delta(\alpha_f)|.$$

When does one have equality?

Zero entropy?

Which principal actions of $\mathbb{H}$ have zero entropy?

Here is an example from Lind-S '15, based on Deninger '11:

$$h(\alpha_{y-x+1/x}) = h(\alpha_{x^2-yx-1}) = h(\alpha_{y^2-xy-1}) = h(\alpha_{y^2-yxz^{-1}-1}) = 0.$$