The Ellis semigroup of a nonautonomous discrete dynamical systems

M. Sanchis[1]

Institut Universitari de Matemàtiques i Aplicacions de Castelló (IMAC), UJI

Dynamical Methods in Algebra, Geometry and Topology
July 4–6, 2018–Udine (Italy)

[1] Joint research with S. García–Ferreira
Centro de Ciencias Matemáticas, Universidad Nacional Autónoma de México (México)

1. Introduction
2. Nonautonomous dynamical systems and the Ellis semigroup
3. Applications of the Ellis semigroup

Definition

Definition

The *Ellis semigroup* of a discrete dynamical system (X, f), denoted by $E(X, f)$, is the closure of $\{f^n : n \in \mathbb{N}\}$ inside of the compact space X^X.

Definition

The *Ellis semigroup* of a discrete dynamical system (X, f), denoted by $E(X, f)$, is the closure of $\{f^n : n \in \mathbb{N}\}$ inside of the compact space X^X.

It well-known that $E(X, f)$ is a compact semigroup whose operation is the composition of functions.

Definition

The *Ellis semigroup* of a discrete dynamical system (X, f), denoted by $E(X, f)$, is the closure of $\{f^n : n \in \mathbb{N}\}$ inside of the compact space X^X.

It well-known that $E(X, f)$ is a compact semigroup whose operation is the composition of functions.

E. Glasner, *Enveloping semigroups in topological dynamics*, Topology Appl. **154** (2007), no. 11, 2344–2363

Theorem

Theorem

If (X, f) is a discrete dynamical system, then

Theorem

If (X, f) is a discrete dynamical system, then

(a)

$$E(X, f) = \{f^p \mid p \in \beta\mathbb{N}\}.$$

Theorem

If (X, f) is a discrete dynamical system, then

(a)

$$E(X, f) = \{f^p \mid p \in \beta\mathbb{N}\}.$$

(b) $E(X, f)$ is a quotient of $\beta\mathbb{N}$.

Definition

Definition

A *nonautonomous discrete dynamical system* is a pair $(X, f_{1,\infty})$

Definition

A *nonautonomous discrete dynamical system* is a pair $(X, f_{1,\infty})$ where X is a compact metric space and $f_{1,\infty}$ is a sequence of continuous functions $(f_n : X \to X)_{n\in\mathbb{N}}$.

Definition

A *nonautonomous discrete dynamical system* is a pair $(X, f_{1,\infty})$ where X is a compact metric space and $f_{1,\infty}$ is a sequence of continuous functions $(f_n : X \to X)_{n\in\mathbb{N}}$.

S. Kolyada and L. Snoha, *Topological entropy of nonautonomous dynamical systems*, Random Comp. Dynam. **4** (1996), no. 2-3, 205–233.

– Given $n \in \mathbb{N}$, the *n-iterate* of a nonautonomous discrete dynamical system $(X, f_{1,\infty})$ is the composition

$$f_1^n := f_n \circ f_{n-1} \circ .... \circ f_2 \circ f_1.$$

– Given $n \in \mathbb{N}$, the *n-iterate* of a nonautonomous discrete dynamical system $(X, f_{1,\infty})$ is the composition

$$f_1^n := f_n \circ f_{n-1} \circ .... \circ f_2 \circ f_1.$$

– If $(X, f_{1,\infty})$ is a nonautonomous discrete dynamical system, then the *orbit* of a point $x \in X$ is the set

$$\mathcal{O}_{f_{1,\infty}}(x) := \{x, f_1^1(x), f_1^2(x), ..., f_1^n(x), ...\}.$$

Definition

Definition The *Ellis semigroup* of a nonautonomous discrete dynamical system $(X, f_{1,\infty})$ is the pointwise closure of the set $\{f_1^n \,|\, n \in \mathbb{N}\}$ inside of the compact space X^X.

This compact space will be denoted by $E(X, f_{1,\infty})$.

$$\beta\mathbb{N} = \{p \mid p \text{ is an ultrafilter on } \mathbb{N}\}$$

$$\beta\mathbb{N} = \{p \mid p \text{ is an ultrafilter on } \mathbb{N}\}$$

The topology of $\beta\mathbb{N}$

$$\beta\mathbb{N} = \{p \mid p \text{ is an ultrafilter on } \mathbb{N}\}$$

The topology of $\beta\mathbb{N}$

$$\bar{A} = \{p \in \beta\mathbb{N} \mid A \in p\}$$

$$\beta\mathbb{N} = \{p \mid p \text{ is an ultrafilter on } \mathbb{N}\}$$

The topology of $\beta\mathbb{N}$

$$\bar{A} = \{p \in \beta\mathbb{N} \mid A \in p\}$$

$$\boxed{\{\bar{A} \mid A \subseteq \mathbb{N}\} \text{ is a base for the topology of } \beta\mathbb{N}}$$

Definition (Bernstein (1970), Furstenberg (1981), Akin (1997))

Definition (Bernstein (1970), Furstenberg (1981), Akin (1997))

Let X be a topological space. A point $x \in X$ is said to be a p-limit point of a sequence $(x_n)_{n\in\mathbb{N}}$ if for all neigboorhood V of x we have

Definition (Bernstein (1970), Furstenberg (1981), Akin (1997))

Let X be a topological space. A point $x \in X$ is said to be a *p-limit point* of a sequence $(x_n)_{n\in\mathbb{N}}$ if for all neigboorhood V of x we have

$$\{n \in \mathbb{N} \mid x_n \in V\} \in p.$$

Definition

Definition

Let p, q be two free ultrafilters. The ultrafilter $p + q$ is defined as

Definition

Let p, q be two free ultrafilters. The ultrafilter $p+q$ is defined as

$$p+q=\{S\subseteq\mathbb{N}\mid\{m\mid\{n\mid m+n\in S\}\in q\}\in p\}.$$

Definition

Definition

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system. For each $p \in \mathbb{N}^*$, we define the function $f_1^p : X \to X$ by

$$f_1^p(x) = p - \lim_{n\to\infty} f_1^n(x),$$

for every $x \in X$.

This function f_1^p is called the p-iterate of the nonautonomous discrete dynamical system $(X, f_{1,\infty})$, for each $p \in \mathbb{N}^*$.

Theorem

Theorem

For every nonautonomous discrete dynamical system $(X, f_{1,\infty})$, we have that

$$E(X, f_{1,\infty}) = \{f_1^p : p \in \beta(\mathbb{N})\}$$

and $f_1^p = p - \lim_{n\to\infty} f_1^n$, for all $p \in \mathbb{N}^*$, in the pointwise topology.

The operation on $E(X, f_{1,\infty})$

The operation on $E(X, f_{1,\infty})$

If $p, q \in \beta(\mathbb{N})$, then we define

$$f_1^p * f_1^q := f_1^{q+p}.$$

The operation on $E(X, f_{1,\infty})$

If $p, q \in \beta(\mathbb{N})$, then we define

$$f_1^p * f_1^q := f_1^{q+p}.$$

Remark. This operation of $E(X, f_{1,\infty})$ does not coincide with the composition of functions.

The operation on $E(X, f_{1,\infty})$

If $p, q \in \beta(\mathbb{N})$, then we define

$$f_1^p * f_1^q := f_1^{q+p}.$$

Remark. This operation of $E(X, f_{1,\infty})$ does not coincide with the composition of functions.

Let $X = [0,1]$ and let $f_n : X \to X$ be defined by $f_n(x) = \frac{1}{n+1}$, for each $x \in X$ and for each $n \in \mathbb{N}$. Then

The operation on $E(X, f_{1,\infty})$

If $p, q \in \beta(\mathbb{N})$, then we define

$$f_1^p * f_1^q := f_1^{q+p}.$$

Remark. This operation of $E(X, f_{1,\infty})$ does not coincide with the composition of functions.

Let $X = [0, 1]$ and let $f_n : X \to X$ be defined by $f_n(x) = \frac{1}{n+1}$, for each $x \in X$ and for each $n \in \mathbb{N}$. Then

$$f_1 * f_1 = f_n^2 = f_2 \circ f_1 = f_2 \text{ and } f_1 \circ f_1 = f_1$$

.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems. Then, the space $E(X, f_{1,\infty})$ is a continuous image of $E(X, f_{k,\infty})$ for each $k \in \mathbb{N}$ with $k \geq 2$.

Theorem
Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems. Then, the space $E(X, f_{1,\infty})$ is a continuous image of $E(X, f_{k,\infty})$ for each $k \in \mathbb{N}$ with $k \geq 2$.

Remark The previous theorem gives us a sequence of continuous surjections:

$$...... \to E(X, f_{k,\infty}) \to ..... \to E(X, f_{2,\infty}) \to E(X, f_{1,\infty}),$$

for every nonautonomous discrete dynamical systems $(X, f_{1,\infty})$.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems such that

$$f_{1,\infty} = (f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, ...)$$

where $1 < k \in \mathbb{N}$ and $f_i : X \to X$ is a continuous functions for each $1 \leq i \leq k$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems such that

$$f_{1,\infty} = (f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, ...)$$

where $1 < k \in \mathbb{N}$ and $f_i : X \to X$ is a continuous functions for each $1 \leq i \leq k$. If $p \in C_j^*$ for some $j < k$,

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical systems such that

$$f_{1,\infty} = (f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, ...)$$

where $1 < k \in \mathbb{N}$ and $f_i : X \to X$ is a continuous functions for each $1 \leq i \leq k$. If $p \in C_j^*$ for some $j < k$, then $f_1^p \in f_j \circ .... \circ f_2 \circ f_1 \circ E(X, f_k \circ ...... \circ f_2 \circ f_1)$

Notation Here, for each $1 < k \in \mathbb{N}$ and $i < k$, we let $C_i = \{n \in \mathbb{N} : n \equiv i \bmod(k)\ \}$.

Definition

Definition

Following Y. Shi (2012), we say that a nonautonomous discrete dynamical systems $(X, \hat{f}_{1,\infty})$ is *induced* by the nonautonomous discrete dynamical systems $(X, f_{1,\infty})$ if there is a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers such that $\hat{f}_1 = f_{k_1} \circ ..... \circ f_1$ and $\hat{f}_n = f_{k_n} \circ ..... \circ f_{k_{n-1}+1}$ for each positive $n \in \mathbb{N}$ bigger than 1.

Proposition

Proposition

Let $(X, \hat{f}_{1,\infty})$ be a nonautonomous discrete dynamical systems *induced* by the nonautonomous discrete dynamical systems $(X, f_{1,\infty})$ via a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers.

Proposition

Let $(X, \hat{f}_{1,\infty})$ be a nonautonomous discrete dynamical systems *induced* by the nonautonomous discrete dynamical systems $(X, f_{1,\infty})$ via a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers. If $k_n + k_m = k_{n+m}$ for all $n, m \in \mathbb{N}$,

Proposition

Let $(X, \hat{f}_{1,\infty})$ be a nonautonomous discrete dynamical systems *induced* by the nonautonomous discrete dynamical systems $(X, f_{1,\infty})$ via a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers. If $k_n + k_m = k_{n+m}$ for all $n, m \in \mathbb{N}$, then $E(X, \hat{f}_{1,\infty})$ is a subsemigroup of $E(X, f_{1,\infty})$.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system. If either

1. the family $\{f_1^n : n \in \mathbb{N}\}$ is equicontinuous; or
2. $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$ and the family $\{\phi \circ f_1^n : n \in \mathbb{N}\}$ is equicontinuous,

then the function $f_1^p : X \to X$ is continuous for each $p \in \mathbb{N}^*$.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. Then $\phi^q \circ f_1^p = f_1^{p+q}$ for every $p \in \mathbb{N}^*$ and for every $q \in \beta(\mathbb{N})$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. Then $\phi^q \circ f_1^p = f_1^{p+q}$ for every $p \in \mathbb{N}^*$ and for every $q \in \beta(\mathbb{N})$. In particular, the Ellis semigroup $E(X, \phi)$ acts on the Ellis semigroup $E(X, f_{1,\infty})^*$.

Definition

Definition

Given a nonautonomous discrete dynamical system $(X, f_{1,\infty})$, for each $p \in \mathbb{N}^*$, we define $E_p(X, f_{1,\infty}) := \{f_1^{p+q} : q \in \beta(\mathbb{N})\}$. Notice that $E_p(X, f_{1,\infty})$ is a subsemigroup of $E(X, f_{1,\infty})$ for every $p \in \mathbb{N}^*$.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. Then $E_p(X, f_{1,\infty})$ is a continuous image of $E(X, \phi)$ for each $p \in \mathbb{N}^*$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. Then $E_p(X, f_{1,\infty})$ is a continuous image of $E(X, \phi)$ for each $p \in \mathbb{N}^*$. As a consequence, $E_p(X, f_{1,\infty})$ is a quotient space of $E(X, \phi)$ for every $p \in \mathbb{N}^*$.

Theorem

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$.

Theorem

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. Then, the function $\phi : X \to X$ has a fixed point in $\omega(x, f_{1,\infty})$ for every $x \in X$.

Corollary

Corollary

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$. If $p \in \mathbb{N}^*$ is an idempotent,

Corollary

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$.
If $p \in \mathbb{N}^*$ is an idempotent,
then $f_1^{q+p}(x)$ is a fixed point of ϕ^p for each $q \in \mathbb{N}^*$ and $x \in X$.

Corollary

Corollary

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$, let $p \in \mathbb{N}^*$ and let $x \in X$.

Corollary

Let $(X, f_{1,\infty})$ be a nonautonomous discrete dynamical system such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $\phi : X \to X$, let $p \in \mathbb{N}^*$ and let $x \in X$.
Then, $f_1^p(x)$ is a periodic point of ϕ iff there is $n \in \mathbb{N}$ such that $f_1^p(x) = f_1^{p+n}(x)$.

Definition

Definition

Two nonautonomous discrete dynamical systems $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ are called *topologically semi-conjugate* if there is a surjective continuous function $h : X \to Y$ such that $g_n(h(x)) = h(f_n(x))$ for every $x \in X$ and for every $n \in \mathbb{N}$. If the function h is a homeomorphism, then we say that they are *topologically conjugate*.

Theorem

Theorem

Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two nonautonomous discrete dynamical systems topologically semi-conjugate via the surjective continuous function $h : X \to Y$. Then, $E(Y, g_{1,\infty})$ is a continuous image of $E(X, f_{1,\infty})$.

Theorem

Let $(X, f_{1,\infty})$ and $(Y, g_{1,\infty})$ be two nonautonomous discrete dynamical systems topologically semi-conjugate via the surjective continuous function $h : X \to Y$. Then, $E(Y, g_{1,\infty})$ is a continuous image of $E(X, f_{1,\infty})$. If h is a homeomorphism, then $E(Y, g_{1,\infty})$ and $E(X, f_{1,\infty})$ are topologically isomorphic.

That's all folks

That's all folks

THANK YOU FOR YOUR ATTENTION