THE AMENABILITY CONJECTURE FOR GROMOV-HYPERBOLIC GROUPS, I

Andrea Sambusetti

(joint work with R. Coulon & F. Dal'Bo – to appear in G.A.F.A.)

DYNAMICAL METHODS IN ALGEBRA, GEOMETRY & TOPOLOGY

July 4-6, Università di Udine

Summary

- history
- beyond amenability: property (T)
- the "easy implication" for hyperbolic groups

1st talk

- main results
- idea of proof:
  - turning the data into a dynamical system
  - a generalization of Kesten–Stadlbauer spectral criterion
- describe in detail the system(s) and their properties (transitivity, visibility)

2nd talk

- History -

Kesten, 1959 G countable group, φ = probability (symmetric, generating) distribution

$M_\varphi : \ell^2(G) \circlearrowleft$ Markov operator for the associated random walk :

G is amenable $\iff$ spectral radius $\rho(M_\varphi) = \limsup_{n\to\infty} \sqrt[n]{P_n(e)} = 1$

$\exists$ G-invariant "mean"

i.e. finitely additive, finite, G-invariant measure

return probability to e after n steps

Examples:
- all f.g. groups of subexp growth
- all solvable groups

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Grigorchuk, Cohen '80 $G = F(S)/N$ finitely generated by S (symmetric)

M_S = Markov operator for the random walk associated to S

$$\rho(M_S) = \frac{\sqrt{e^{\omega_{F(S)}}}}{1+e^{\omega_{F(S)}}}\left(\frac{\sqrt{e^{\omega_{F(S)}}}}{e^{\omega_N}} + \frac{e^{\omega_N}}{\sqrt{e^{\omega_{F(S)}}}}\right)$$

where : $\omega_{F(S)}$ = exp. growth rate (with respect to S) $= \ln(2|S|-1)$

ω_N = exp. growth rate of $N \triangleleft (G,S)$

more generally : $\omega(G \curvearrowright X) = \limsup\limits_{R\to\infty} \frac{1}{R} \cdot \ln \#\{g \mid d(x,gx) \leq R\}$

entropy, critical exponent ($= \omega_G$ when $G \curvearrowright X$ is understood)

- History -

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where: $\omega_{F(S)}$ = exp. growth rate (with respect to S) = $\ln(2|S|-1)$

ω_N = exp. growth rate of $N \triangleleft (G,S)$

COROLLARY $G = F(S)/N$ amenable $\Longleftrightarrow$ $\omega_{F(S)} = \omega_N$

Brooks '84 (first escape from the realm of abstract groups)

$\hat{M} \to M$ normal covering of a compact Riemannian manifold,

Γ = automorphism group, i.e. $M = \hat{M}/\Gamma$:

bottom of the spectrum of Hodge-Laplace Δ

$$\Gamma \text{ amenable} \Longrightarrow \lambda_0(\hat{M}) = \lambda_0(M)$$

and $\Longleftarrow$ holds for "special" M

e.g.: M real hyperbolic manifold

Now consider:

- a compact hyperbolic manifold $M = \mathbb{H}^n/G$ ($\longrightarrow \omega_G = n-1 =$ "$\omega_{\mathbb{H}^n}$")
- a normal covering $\hat{M} = \mathbb{H}^n/N \longrightarrow M = \mathbb{H}^n/G$ with $N \triangleleft G$

coupling with Sullivan's formula for Kleinian groups N of $\mathbb{H}^n$:

$$\lambda_0(\mathbb{H}^n/N) = \begin{cases} \lambda_0(\mathbb{H}^n) & \text{if } \omega_N \leq \frac{n-1}{2} \\ \omega_N(n-1-\omega_N) & \text{if } \omega_N \geq \frac{n-1}{2} \end{cases}$$

(here ω is w.r.to the action on $\mathbb{H}^n$)

Brooks '84

$\hat{M} \to M$ normal covering of a compact Riemannian manifold,
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Now consider:

- a compact hyperbolic manifold $M = \mathbb{H}^n/G$ ($\longrightarrow \omega_G = n-1 =$ "$\omega_{\mathbb{H}^n}$")
- a normal covering $\hat{M} = \mathbb{H}^n/N \longrightarrow M = \mathbb{H}^n/G$ with $N \lhd G$

COROLLARY G = lattice of $\mathbb{H}^n$, $N \lhd G$ with $\omega_N \geq \frac{n-1}{2}$:

$$\omega_N = \omega_G \iff \lambda_0(\mathbb{H}^n/N) = \lambda_0(\mathbb{H}^n/G) \iff G/N \text{ amenable}$$

Roblin, 2003 G discrete group of isometries of X = CAT(-1)-space

N normal in G : G/N amenable $\Rightarrow \omega_G = \omega_N$

AMENABILITY CONJECTURE

for any normal sbgp $N \lhd G$ → "negatively curved" group
e.g. cocompact CAT(-1)-group

G/N amenable $\overset{?}{\Longleftrightarrow} \omega_G = \omega_N$

Possibly: for Gromov hyperbolic groups ?

A recent substantial step forward (details later):

Stadlbauer's spectral Amenability Criterion, 2013

(extending Kesten) ⇝ settling the conjecture for:

- "essentially free" Kleinian groups [Stadlbauer]
- "co-co-co" groups of pinched, negatively curved Hadamard manifolds [Dougall – Sharp, 2014]

– Beyond amenability –

there exist some (hyperbolic, negatively curved) groups G whose subgroups N cannot have ω_N even close to ω_G :

Corlette '90 G = lattice of $\mathbb{H}^n(\mathbb{K})$, where $\mathbb{K}$ = quaternions or Cayley numbers :

$\exists\, \varepsilon > 0$ such that $N < G \Rightarrow \omega_N \leq \omega_G - \varepsilon$ (unless $\omega_N = \omega_G$ and $|G/N| < \infty$)

Notice : $\mathrm{Isom}(\mathbb{H}^n(\mathbb{K}))$ and such G have KAZHDAN Property (T)

AIMS of our work :

1) translate the amenability problem for hyperbolic groups in spectral terms using a "CANONICAL" dynamical system

hopefully $\rightarrow$ clarifying & extending existing results into a general unified setting (groups with word metrics, groups acting on CAT(-1)-spaces, π_1 of manifolds ...)

2) try to enlighten the relation with property (T) and extend Corlette's hyper-rigidity result to hyp. groups satisfying (T)

– Warm-up: the "easy implication" for hyperbolic groups –

Theorem (C-D-S, 2017) Let G be a hyperbolic group:

$$\boxed{\text{if } \exists \text{ subgroup } H < G \text{ is co-amenable} \implies \omega_H = \omega_G}$$

not necessarily normal

works more generally for $G \overset{\text{coco}}{\curvearrowright} X$ hyp. space

– AMENABILITY of an action $G \curvearrowright X$: X admits a G-invariant mean

– $H < G$ is co-amenable if $X = H\backslash G \curvearrowleft G$ is amenable

(G is amenable $\iff$ $G \curvearrowright \mathcal{G}(G,S)$ is amenable

$H \lhd G$, H co-amenable in G $\iff$ G/H amenable)

CHARACTERIZATION of amenability in terms of the unitary regular representation

a transitive action $G \curvearrowright X$ of a countable group is amenable

$\iff$ $\lambda : G \to \mathcal{U}(\ell^2(X))$ ALMOST HAS INVARIANT VECTORS

i.e. $\forall S \subset G$ finite, $\forall \varepsilon$ $\exists v_\varepsilon \in \ell^2(X) - 0$, such that $\|s v_\varepsilon - v_\varepsilon\| \leq \varepsilon \|v_\varepsilon\|$ $\forall s \in S$

Notice: $G \overset{\lambda}{\curvearrowright} \ell^2(X)$ has a TRUE G-invariant vector $\iff$ X IS FINITE!

KAZHDAN PROPERTY (T): a countable group G has (T) if $\exists$ finite $S \subset G$ and $\exists\, \varepsilon > 0$ such that for any unitary representation $\rho_{\mathcal{H}} : G \to \mathcal{U}(\mathcal{H})$

$\rho_{\mathcal{H}}$ has (S, ε)-invariant vector v_ε $\Longrightarrow$ $\rho_{\mathcal{H}}$ has a true S-invariant vector v

EX: all finite & compact Lie groups, simple Lie groups with $\mathbb{R}$-rank ≥ 2 with finite center ($Sl_{n\geq 3}(\mathbb{K})$, $Sp_{2n\geq 4}(\mathbb{K})$, $Sp(n\geq 2, 1)$ ...) and their lattices

"orthogonal" to amenability

Fact: for countable G, amenable + (T) $\Longleftrightarrow$ finite !

Some astonishing topological-geometric consequences of property (T):

- finite generation of π_1 for lattices of certain loc. sym. manifolds
- Corlette's hyper-rigidity result
- Property (FA) of Serre for actions on trees, and property (FH) for affine isometries of Hilbert spaces
- Construction of EXPANDERS
- local conjugacy rigidity of smooth isometric actions on cpt. Riemannian manifolds ...

Theorem (C-D-S, 2017) Let G be a hyperbolic group:

$$\boxed{\text{If } \exists \text{ subgroup } H < G \text{ is } \underline{\text{co-amenable}} \Rightarrow \omega_H = \omega_G}$$

* Sketch of proof: let G δ-hyperbolic + assume $X = H\backslash G \curvearrowleft G$ amenable

$S(R) = \delta$-sphere of radius $R = B_G(e,R) - B_G(e,R-\delta)$

$\mu_R =$ uniformly distributed measure on $S(R)$

We use Kesten's Criterion: $\rho(M_R) = 1 \ \forall R$

For hyperbolic groups we can only prove:

$$\boxed{\limsup_{R\to\infty} \sqrt[R]{\rho(M_R)} = e^{-\max\left\{\omega_G - \omega_H,\ \frac{\omega_G}{2}\right\}}}$$

if $H\backslash G \curvearrowleft G$ amenable $\Rightarrow \rho(M_R) = 1 \ \forall R$

$\Rightarrow$ either $\omega_G = 0$ (G has subexp. growth) or $\omega_G = \omega_H$ $\square$

But it does not work for the converse $\Leftarrow$!!

Theorem (C-D-S, 2017) Let G be a hyperbolic group:

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For hyperbolic groups we can only prove:

$$\boxed{\limsup_{R\to\infty} \sqrt[R]{\rho(M_R)} \leqslant e^{-\max\left\{\omega_G - \omega_H,\ \frac{\omega_G}{2}\right\}}}$$

enough!

$\to$ if $H\backslash G \curvearrowleft G$ amenable $\Rightarrow \rho(M_R) = 1 \quad \forall R$

$\Rightarrow$ either $\omega_G = 0$ (G has subexp. growth) or $\omega_G = \omega_H$ $\square$

G δ-hyperbolic, $X = H\backslash G$

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$\mu_R =$ uniformly distributed measure on $S(R)$

$M_R : \ell^2(X) \circlearrowleft$ Markov operator associated to μ_R

$\rho(M_R) = \limsup_{n\to\infty} \sqrt[n]{P_R(H,n)}$ spectral radius of M_R

$$\boxed{\limsup_{R\to\infty} \sqrt[R]{\rho(M_R)} \overset{?}{\leq} e^{-\max\left\{\omega_G - \omega_H,\ \frac{\omega_G}{2}\right\}}}$$

We need :

ESTIMATE on any δ-hyperbolic group G $\exists C>0$ such that

$$\mu_R^{*n}(g) \leq C^n\left(\frac{R}{\delta}+1\right)^n e^{-\frac{\omega_G}{2}(nR+|g|)} \quad \forall g\in G \quad (\text{if } \delta \gg 0)$$

⚠ Rough! only the exponential decay in nR matters

$\to$ only depending on $|g|$

"uniform upper isotropy"

G δ-hyperbolic, $X = H \backslash G$

$S(R) =$ δ-sphere of radius $R = B_G(e,R) - B_G(e,R-\delta)$

$\mu_R =$ uniformly distributed measure on $S(R)$

$M_R : \ell^2(X) \circlearrowleft$ Markov operator associated to μ_R

$\rho(M_R) = \limsup_{n\to\infty} \sqrt[n]{p_R(H,n)}$ spectral radius of M_R

$$\boxed{\limsup_{R\to\infty} \sqrt[R]{\rho(M_R)} \overset{?}{\leq} e^{-\max\left\{\omega_G - \omega_H,\ \frac{\omega_G}{2}\right\}}}$$

We need :

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$$\Rightarrow p_R(H,n) = \sum_{k\geq 0} \sum_{h\in H\cap S(k\delta)} \mu_R^{*n}(h)$$

plug here

$$\leq \sum_{k\geq 0} C^n\left(\frac{R}{\delta}+1\right)^n e^{-\frac{\omega_G}{2}(nR+|g|)} \underbrace{\left|H\cap S(k\delta)\right|}_{\asymp_C\, e^{(\omega_H+\varepsilon)k\delta}}$$

and now just compute ! $\square$

Proof of the estimate for $\mu_R^{*n}(g) = \sum\limits_{\substack{(u_1 \cdots u_n) \in S(R)^n \\ u_1 \cdots u_n = g}} \mu_R(u_1) \cdots \mu_R(u_n)$

by induction on n :

$\{(u_1 \cdots u_n) \in S(R)^n,\ u_1 \cdots u_n = g\} \rightarrow$ the set $W_R^n(g)$

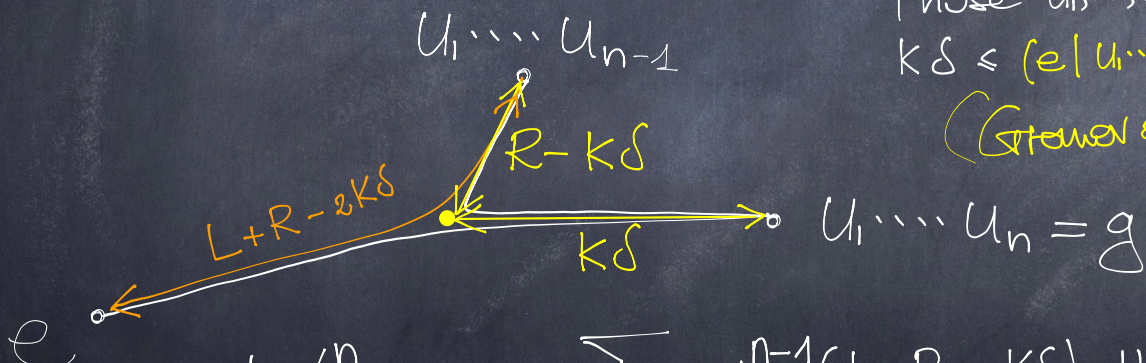
for any $|g| = L$ $\quad \# W_R^n(g) \overset{?}{\leq} w_R^n(L) = c^n \left(\frac{R}{\delta}+1\right)^n e^{-\frac{\omega_G}{2}(nR+L)}$

partition $W_R^n(g) = \coprod\limits_K \underbrace{W_R^n(g, K)}$

those $u_1, \dots, u_n$ such that

$K\delta \leq (e \mid u_1 \cdots u_{n-1})_g \leq (K+1)\delta$

(Gromov product)



$$\# W_R^n(g) \leq \sum_K w_R^{n-1}(\underbrace{L+R-2K\delta}_{L' < L})\, \underbrace{\# S(K\delta)}_{\asymp e^{\omega_G K\delta}}$$

now apply the induction for $w_R^{n-1}(L')$ $\square$

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- main results
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  - a generalization of Kesten–Stadlbauer spectral criterion
- the system(s) in detail and their properties (transitivity, visibility)

2nd talk

- Main results -

<u>Theorem 1</u> [C-D-S] (G,S) δ-hyperbolic group with finite generating set:

$\exists$ sbgrp $H<G$ has $\omega_G=\omega_H$ $\iff$ H co-amenable in G

($\Leftarrow$) yesterday

<u>Theorem 2</u> [C-D-S] (G,S) δ-hyperbolic group with property (T):

$\exists\ \varepsilon=\varepsilon(G,S)>0$ such that for any sbgrp $H<G$ it holds
$\omega_H<\omega_G-\varepsilon$, unless H has finite index in G.

<u>COROLLARY</u> Let X be a Hadamard manifold with $K_X\leq -1$ admitting lattices, and whose isometry group $Is(X)$ possesses (T):
$\exists\ \varepsilon=\varepsilon(X)$ such that <u>any</u> discrete subgroup H of $Is(X)$ satisfies $\omega_H<\omega_X-\varepsilon$, or it is itself a lattice.

– Strategy of proof –

1) TRANSLATE $\begin{cases} (G,S) \rightsquigarrow \text{dynamical system representing the "geodesic flow" of } G \\ \qquad\qquad + \text{classical transfer operator } \mathcal{L} \text{ describing the evolution} \\ H<G \rightsquigarrow \text{new transfer operator } \mathcal{L}_H \text{ attached to } Y = H\backslash G \curvearrowleft G \end{cases}$

and the GAP $\omega_G - \omega_N$ into a spectral gap $\rho(\mathcal{L}) - \rho(\mathcal{L}_H)$

Gromov: there exist

- (Σ, σ) subshift of finite type $\rightsquigarrow$ "space of directions" of $C(G,S)$ at e

$\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ where A = finite alphabet

$\Sigma \subset A^{\mathbb{N}}$ (closed) σ-invariant "admissible sequences"

$+ \ \exists F$ = finite set of words $\subset A^n$

which "detects" the admissible sequences:

$x \in \Sigma$ iff every subword of length n of x is in F

– Strategy of proof –

1) TRANSLATE
- $(G,S) \rightsquigarrow$ dynamical system representing the "geodesic flow" of G + classical transfer operator $\mathcal{L}$ describing the evolution
- $H < G \rightsquigarrow$ new transfer operator $\mathcal{L}_H$ attached to $Y = H\backslash G \circlearrowleft G$

and the GAP $\omega_G - \omega_N$ into a spectral gap $\rho(\mathcal{L}) - \rho(\mathcal{L}_H)$

Gromov: there exist

- (Σ, σ) subshift of finite type $\rightsquigarrow$ "space of directions" of $\mathcal{C}(G,S)$ at e $\sim (UM, \varphi^t_{geo})$ unitary tangent bundle of cpt. Riemannian manifold
- (Σ_G, σ_G) extension of (Σ, σ) by evaluation map $\vartheta : \Sigma \to G$ $\sim (\widetilde{UM}, \tilde{\varphi}^t_{geo})$

formally
$$\Sigma_G = \Sigma \times G$$
$$\sigma_G(x, g) = (\sigma x, g \cdot \vartheta(x))$$

good to know: The geodesic trajectory on $\mathcal{C}(G,S)$ from e determined by $x \in \Sigma$ is given by the (infinite) sequence $\vartheta_n(x) := \vartheta(x) \cdot \vartheta(\sigma x) \cdots \vartheta(\sigma^{n-1} x) \quad (n \geq 0)$

they have (unexpectedly?) "good properties" for counting

– Strategy of proof –

1) TRANSLATE
$(G,S) \rightsquigarrow$ dynamical system representing the "geodesic flow" of G
+ classical transfer operator $\mathcal{L}$ describing the evolution

$H<G \rightsquigarrow$ new transfer operator $\mathcal{L}_H$ attached to $Y = H\backslash G \circlearrowleft G$

and the GAP $\omega_G - \omega_N$ into a spectral gap $\rho(\mathcal{L}) - \rho(\mathcal{L}_H)$

Gromov: there exist

- (Σ, σ) subshift of finite type $\rightsquigarrow$ "space of directions" of $C(G,S)$ at e
- (Σ_G, σ_G) extension of (Σ, σ) by evaluation map $\theta : \Sigma \to G$
- $\mathcal{L} : H^\infty_\alpha(\Sigma, \mathbb{C}) \circlearrowleft$ classical Ruelle's transfer operator $\mathcal{L}\varphi(x) = \sum_{\sigma x' = x} e^{-\omega_G} \varphi(x')$

potential $F(x')$

- $\mathcal{L}_\lambda : H^\infty_\alpha(\Sigma, \ell^2(Y)) \circlearrowleft$ "twisted" transfer operator $\mathcal{L}_H \varphi(x) = \sum_{\sigma x' = x} e^{-\omega_G} \underbrace{\theta(x') \cdot \varphi(x')}$

unitary left regular representation $\lambda : G \circlearrowleft \ell^2(Y)$

PROPOSITION (growth gap Vs spectral gap)

$$\boxed{\omega_G - \omega_H < \Delta \Rightarrow \rho(\mathcal{L}) - \rho(\mathcal{L}_\lambda) < \eta(\Delta) \xrightarrow{\Delta \to 0} 0}$$

– Strategy of proof –

2) QUANTITATIVE VERSION of Stadlbauer amenability criterion

Theorem [C-D-S]

Let $Y \circlearrowleft G$ action of a f.g. group on a countable set and assume:

- (Σ, σ) subshift of finite type TOPOLOGICALLY TRANSITIVE
  $\Downarrow$
  - there exists a dense orbit
  - $\forall\, U, V$ open sets $\neq \emptyset$
    $\exists\, n$ such that $\sigma^n(U) \cap V \neq \emptyset$
  - for any admissible words u, v
    $\exists\, w_0$ such that $u w_0 z$ is admissible

– Strategy of proof –

2) QUANTITATIVE VERSION of Stadlbauer amenability criterion

Theorem [C-D-S]

Let $Y \circlearrowleft G$ action of a f.g. group on a countable set and assume:

- (Σ, σ) subshift of finite type TOPOLOGICALLY TRANSITIVE
- (Σ_G, σ_G) extension of (Σ, σ) by a (locally constant) evaluation map $\theta : \Sigma \to G$ WITH THE VISIBILITY PROPERTY

means that "the flow visits almost all G"

i.e. $\exists$ finite subset $B \subset G$ such that

$\forall g \in G \; \exists x \in \Sigma$ and $u, v \in B$: $u \theta_n(x) v = g$

(where $\theta_n(x) := \theta(x) \cdot \theta(\sigma x) \cdots \theta(\sigma^{n-1} x)$)

– Strategy of proof –

2) QUANTITATIVE VERSION of Stadlbauer amenability criterion

Theorem [C-D-S]

Let $Y \circlearrowleft G$ action of a f.g. group on a countable set and assume:

- (Σ, σ) subshift of finite type TOPOLOGICALLY TRANSITIVE
- (Σ_G, σ_G) extension of (Σ, σ) by a (locally constant) evaluation map $\vartheta : \Sigma \to G$ WITH THE VISIBILITY PROPERTY
- $\mathcal{L} : H^\infty_\alpha(\Sigma, \mathbb{C}) \circlearrowleft$ transfer operator w.r.to a potential F (with $\|F\|^\infty_\alpha < \infty$)
- $\mathcal{L}_\lambda : H^\infty_\alpha(\Sigma, \ell^2(Y)) \circlearrowleft$ twisted transfer operator by $G \overset{\lambda}{\curvearrowright} \ell^2(Y)$

For every finite $S \subset G$, $\forall \varepsilon > 0$ there exists $\eta = \eta(\varepsilon) > 0$ such that

$$\eta(\varepsilon) \xrightarrow[\varepsilon \to 0]{} 0$$

$$\rho(\mathcal{L}_\lambda) > (1-\eta)\,\rho(\mathcal{L}) \Rightarrow \lambda \text{ has a } (S, \varepsilon)\text{-invariant vector}$$

Conclusion : if we can prove that

+ the "geodesic flow" (Σ, σ) of G is top transitive
+ (Σ_G, σ_G) has the visibility property

then the condition $\omega_G - \omega_H < \Delta \overset{1)}{\Longrightarrow} \rho(\mathcal{L}) - \rho(\mathcal{L}_\lambda) < \eta(\Delta)$

$\overset{2)}{\Longrightarrow} \lambda : G \to \mathcal{U}(\ell^2(H \backslash G))$ has $(S, \varepsilon(\Delta))$ invariant vectors

- So $\omega_G = \omega_H \longrightarrow \lambda$ almost has invariant vectors and so H is co-amenable in G
- and if G has (T) $\longrightarrow$ ω_H cannot be arbitrarily close to ω_G

(OTHERWISE : for the (S, ε) given by property (T) of G,
choose $H < G$ with ω_H sufficiently close to ω_G
$\xrightarrow{1)+2)} \exists$ (S, ε)-invariant vector for $\lambda : G \to \mathcal{U}(\ell^2(H \backslash G))$
$\underset{(T)}{\Longrightarrow} \exists$ true invariant vector for $\lambda \Longrightarrow H \backslash G$ is finite)

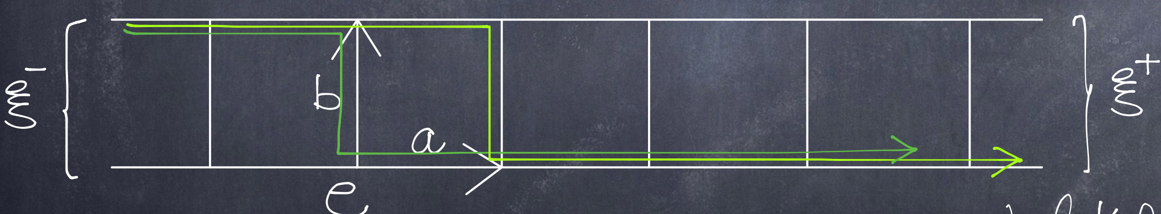
– A closer look to our dynamical system –

(G,S) δ-hyperbolic group $\rightsquigarrow X = \mathcal{C}(G,S)$

$\mathcal{G}X = \left\{ \begin{array}{l} \text{parameterized} \\ \text{bi-infinite} \\ \text{geodesics of } X \end{array} \right\}$, $\varphi^t \gamma(\cdot) = \gamma(\cdot + t)$ $\quad d(\gamma,\gamma') = \int_{-\infty}^{+\infty} e^{-|t|} d(\gamma(t),\gamma'(t))\,dt$

far too pathological $\left(\begin{array}{l} \text{infinitely many geodesics between points at } \infty \\ \text{unboundedly many between points in } X \end{array} \right)$

ex $G = \mathbb{Z} \times \mathbb{Z}_2$, $S = \{a, b\} \rightsquigarrow \partial X = \{\xi^-, \xi^+\}$ Gromov boundary



Remember:

we look for a system which is
+ subshift of finite type (Σ, σ)
+ topologically transitive
+ evaluation $\theta : \Sigma \to G$
with visibility property

two constructions of Gromov's :

1) the "reduced geodesic flow"

2) the "horoflow"

1) the "reduced geodesic flow"

$\mathcal{G}_{red}X = \mathcal{G}X/\sim$ $\quad \gamma \sim \gamma'$ iff $\gamma^+ = \gamma'^+$ and $\gamma^- = \gamma'^-$

contract all geodesics with same endpoints and reparameterize coherently (φ^t_{red})

- $\mathcal{G}_{red}X$ is a proper geodesic space $\simeq \partial X \times_\Delta \partial X \times \mathbb{R}$
- $\mathcal{G}X \xrightarrow{\pi} \mathcal{G}_{red}X$ surjective, quasi-isometries
- $(\mathcal{G}_{red}X, \varphi_{red}) \circlearrowleft G$ commutes with the reparametrized flow

<u>PROPOSITION</u> $(\mathcal{G}_{red}X, \varphi_{red})/G$ is topologically transitive

* proof: copy from the case of manifolds with $K \leq -1$

BUT: <u>no known coding</u> (it is not a subshift of finite type)

2) the "horoflow"

$\partial_{hor} X$ = horoboundary

$\partial^{\mathbb{Z}}_{hor} X = \{$ integral horofunctions $\}$

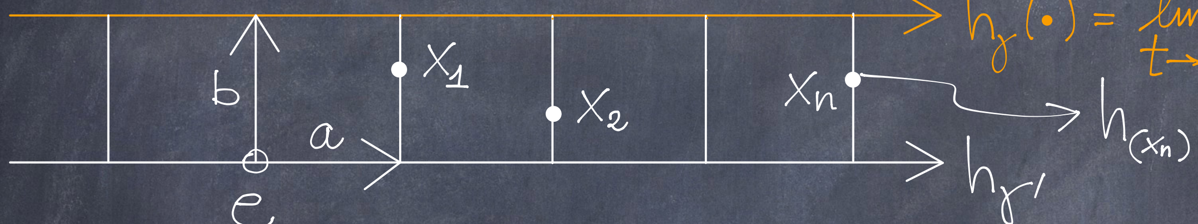
fix $x_0 \in X \xrightarrow{i} Lip(X)$ by its Busemann cocycle

$$x \mapsto b_x(\cdot) = x_0x - x\cdot$$

$$\partial_{hor} X := \overline{i(X)}^{Lip(X)} - i(X)$$

$h(\cdot)$ horofunction $= \lim_n x_0x_n - x_n\cdot$

$$h_\gamma(\cdot) = \lim_{t\to+\infty} x_0\gamma(t) - \gamma(t)\cdot$$



the horoboundary has a flow by gradient lines

2) the "horoflow"

$\partial_{hor} X$ = horoboundary

$\partial^{\mathbb{Z}}_{hor} X = \{$ integral horofunctions $\}$

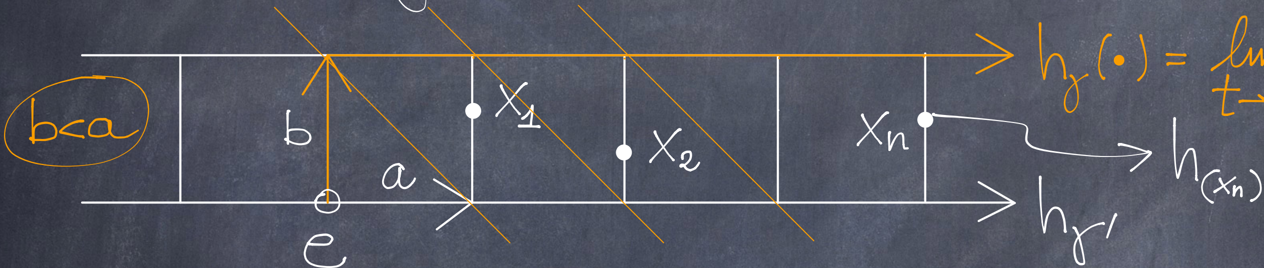
fix $x_0 \in X \overset{i}{\hookrightarrow} Lip(X)$ by its Busemann cocycle

$x \mapsto b_x(\cdot) = x_0x - x\cdot$

$\partial_{hor} X := \overline{i(X)}^{Lip(X)} - i(X)$

$h(\cdot)$ horofunction $= \lim_n x_0x_n - x_n\cdot$

$h_\gamma(\cdot) = \lim\limits_{t \to +\infty} x_0\gamma(t) - \gamma(t)\cdot$



the horoboundary has a flow by gradient lines

fix some lexicographic order on S :

for any horofunction h and any point x

$\exists!$ geodesic γ satisfying $h(\gamma(t)) - h(\gamma(s)) = t - s$

minimal for the lexicographic order

= the gradient line $\gamma_{h\,x}$ of h from x

2) the "horoflow"

much more elaborate construction

$\partial_{hor} X$ = horoboundary

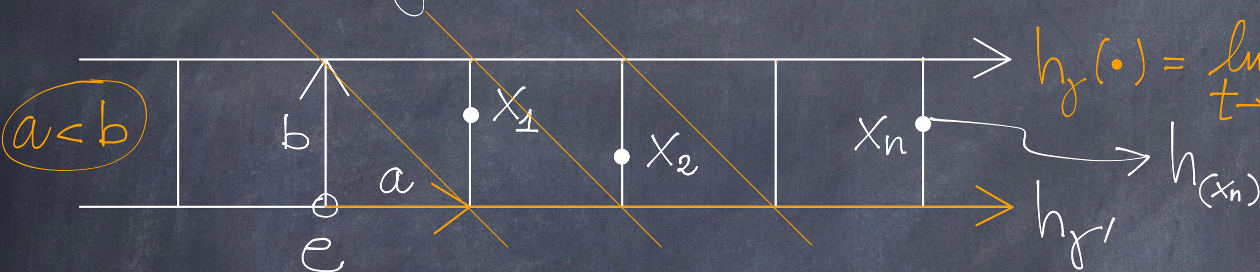
$\partial^{\mathbb{Z}}_{hor} X = \{$ integral horofunctions $\}$

fix $x_0 \in X \overset{i}{\hookrightarrow} Lip(X)$ by its Busemann cocycle

$x \mapsto b_x(\cdot) = x_0 x - x\cdot$

$\partial_{hor} X := \overline{i(X)}^{Lip(X)} - i(X)$

$h(\bullet)$ horofunction $= \lim_n x_0 x_n - x_n \bullet$



$h_\gamma(\bullet) = \lim_{t \to +\infty} x_0\gamma(t) - \gamma(t)\bullet$

the horoboundary has a flow by gradient lines

$G \curvearrowright \mathcal{G}^{\mathbb{Z}}_{hor} X = \{ (h, \gamma) \mid h \in \partial^{\mathbb{Z}}_{hor} X,\ \gamma \in \mathcal{G} X$ minimal integral gradient line of $h \}$

$G \curvearrowright \mathcal{G}_{hor} X = \{$ the same, but with $\gamma : \mathbb{R} \to X \}$

discrete & continuous horoflow φ^t_{hor}

2) the "horoflow" (continued)

Call $\sigma(h) = 1^{st}$ vertex of minimal gradient line $\gamma_{h,e}$ of h from e

Theorem (Gromov, Coornaert - Papadopoulos)

- $(\mathcal{G}^{\mathbb{Z}}_{hor}X, \varphi^n_{hor})/G = (\partial^{\mathbb{Z}}_{hor}X, T_{hor})$ where $T_{hor}(h) = \sigma(h)^{-1}\cdot h$
  and is conjugated to a subshift of finite type (Σ, σ)
- $(\mathcal{G}^{\mathbb{Z}}_{hor}X, \varphi^n_{hor})/G$ is conjugated to $Susp_1(\Sigma, \sigma)$

the GOOD: it is a subshift of finite type
the BAD: it is NOT top. transitive

<u>Remark</u>: $(\partial^{\mathbb{Z}}_{hor}X, T_{hor})$ is described by a finite graph $\Gamma(\Sigma, \sigma)$
we can take one irreducible component $\Gamma^{irr} \rightsquigarrow$ transitivity
but (Σ^{irr}_1, σ) might miss lot of G! (~~visibility~~)

IDEA : $\exists$ map $(\mathcal{G}_{hor}X, \varphi^t_{hor}) \xrightarrow{\pi} (\mathcal{G}_{red}X, \varphi^t_{red})$

$(h,\gamma) \longmapsto [\gamma]$

Subshift mod G

transitive mod G

- surjective, quasi-isometric
- G-equivariant
- orbit-preserving (but NOT time-preserving)

- use top. transitivity of φ^t_{red} to find a dense orbit $\varphi^t_{red}[\gamma_0]$ mod G
- use surjectivity of π to lift it to $\varphi^s_{hor}(h_0,\gamma_0) \in \mathcal{G}^{\mathbb{Z}}_{hor}X$ mod G
- use quasi-isometry + hyp. geometry to show that $\varphi^s_{hor}(h_0,\gamma_0)$ visits almost all G

$\rightsquigarrow$ visibility $\square$