INTRINSIC VALUATION ENTROPY

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Abstract. The intrinsic entropy for endomorphisms of modules over a non-discrete archimedean valuation domain R is presented, which extends the analogous notion for endomorphisms of Abelian groups, using the natural non-discrete length function introduced by Northcott and Reufel for such a category of modules. With the aid of new techniques suitable for the non-discrete setting, we prove that this notion of entropy is a length function for the category of R[X]-modules, it satisfies a suitably adapted version of the Yuzvinski Formula, and it is essentially the unique invariant for Mod(R[X]) with these properties.

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SUMMARY

1. Algebraic entropies induced by length functions

2. Length functions for modules over valuation domains

3. Valuation entropy ent_V and intrinsic valuation entropy $\widetilde{\mathrm{ent}}_V$

4. A new tool: the anti-trajectories

5. $\widetilde{\mathrm{ent}}_V$ is a length function on Mod(R[X])

6. Intrinsic Yuzvinsky Formula and Uniqueness theorem

1. Algebraic entropies induced by length functions

Given the category of modules over a unitary ring R, a length function is a tool assigning to each module a size, measured by non-negative real numbers or the symbol ∞, an axiomatic generalization of the classical notion of composition length

Northcott and Reufel in 1965 gave the following

DEFINITION. Let R be a commutative ring. A function $L: \mathrm{Mod}(R) \to \mathbf{R}^* = \mathbf{R}_{\geq 0} \cup \{\infty\}$ is a length function if it satisfies the following two conditions:

(A) given an exact sequence in Mod(R): $0 \to A \to B \to C \to 0$, $L(B) = L(A) + L(C)$;

(UC) for every module M, $L(M) = \sup \{L(F) : M \geq F \text{ fin. gen.}\}$

Once we have this tool to measure the size of modules, we can define the associated L-entropy of an endomorphism of an R-module M.

Denote by $\mathcal{F}(M)$ the family of the submodules H of M such that $L(H) < \infty$.

DEFINITIONS. 1) Let R be a ring and $\varphi : M \to M$ an endomorphism of an R-module. For every $H \in \mathcal{F}(M)$ set $T_n(\varphi,H) = H + \varphi H + \ldots + \varphi^{n-1}H$ and

$$\mathrm{ent}_L(\varphi,H) = \lim_{n\to\infty} L(T_n(\varphi,H)) / n.$$

This limit exists finite and it is equal to $\inf_n \{L(T_n(\varphi,H))/n\}$, by Fekete's Lemma; it is called the *L-entropy of φ with respect to H*.

2) The real number or symbol ∞:

$$\mathrm{ent}_L(\varphi) = \sup\{ \mathrm{ent}_L(\varphi,H) : H \in \mathcal{F}(M) \}$$

is the *L-entropy* of φ.

REMARK. The L-entropy may be defined for functions L satisfying only:

- $L(0) = 0$
- $M \cong N \Rightarrow L(M) = L(N)$
- $N \leq M \Rightarrow L(M/N) \leq L(M)$
- $L(M + M') \leq L(M) + L(M')$.

Such a function is a *sub-additive invariant.*
The associated L-entropy does not satisfy fundamental results holding for length functions, e.g., the Addition Theorem and the Uniqueness Theorem.

We will always consider commutative rings R. A length function L is

- *non-trivial* if $L(M) > 0$ for some module M;
- *of 0/∞-type* if it takes only values 0 and ∞;
- *faithfull* if $L(M) = 0$ implies $M = 0$;
- *discrete* if its finite values form a discrete (unbounded) subset of the reals.

EXAMPLES

1) log | - | for Mod($\mathbb{Z}$)

it is a discrete faithfull length function with associate entropy the first algebraic entropy defined by Adler-Konheim-McAndrew in 1965 and deeply investigated by Dikranjan-Goldsmith-S-Zanardo in 2009 and elsewhere.

2) rk_R for Mod(R), where R denotes a commutative integral domain

it is the well-known notion of rank, a discrete non-faithfull length function.

3) the composition length ℓ for modules over a commutative ring

it is discrete and faithfull.

4) the length function L_V associated with the valuation v of an archimedean valutain domain R, defined by setting $L(R/I) = \inf\{v(a) : a \in I\}$ for all $I \leq R$

it is non-discrete if R is not a DVR

Non-discrete length functions offer a far-reaching generalization of composition length, evaluating the size of modules with real numbers, not only with positive integers. The techniques used for the L-entropy when L is discrete cannot be applied in general when L is non-discrete.

The main result for the L-entropy, the Addition Theorem (AT), was proved over arbitrary rings for L non-discrete by S-Virili for the category of locally L-finite modules (analogous of torsion Abelian groups), introducing new techniques.

Since AT holds for the whole category Mod(Z) for the more general notion of intrinsic entropy, it is natural to try to extend this notion to arbitrary rings R, having at disposal a length function L, also for L non-discrete. The first case to be investigated is that in Example 4), most important for valuation domains, as we will see.

2. Length functions for modules over valuation domains

Northcott and Reufel classified all length functions over a valuation domain R. Excluding the trivial length function which sends all modules to 0, characterized by the property that $L(R) = 0$, length functions L may be distinguished as follows:

♦ if $0 < L(R) < \infty$, then L is the rank function, up to a positive real multiple;

♦♦ if $L(R) = \infty$, then L may be of three different types:

0/∞-TYPE RANK-TYPE VALUATION-TYPE

Since L is completely determined by its action on the cyclic modules R/I, it is enough to look at the values $L(R/I)$ for $0 \leq I \leq R$.

$0/\infty$-TYPE

(I) There is a prime ideal P such that:

$$L(R/I) = 0 \text{ for } P < I \quad \text{and} \quad L(R/I) = \infty \quad \text{for } I \leq P, \qquad \text{or}$$

(II) there is an idempotent prime ideal $P = P^2$ such that:

$$L(R/I) = 0 \text{ for } P \leq I \quad \text{and} \quad L(R/I) = \infty \quad \text{for } I < P.$$

RANK-TYPE

There is an idempotent prime ideal $P = P^2$ and a positive real number λ such that:

$$L(R/I) = 0 \quad \text{if} \quad P < I, \qquad L(R/P) = \lambda, \qquad L(R/I) = \infty \quad \text{if} \quad I < P.$$

It follows that $L(M) = \lambda \cdot rk_{R/P}(M)$ if $PM = 0$, $L(M) = \infty$ otherwise.

If $P = 0$, then $L(M) = \lambda \cdot rk_R(M)$ for every module M, already considered above.

VALUATION-TYPE

Let R be a valuation domain with field of quotients Q and value group $\Gamma = Q^{x}/U(R)$ and let $v : Q \to \Gamma \cup \{\infty\}$ be the associated valuation.

Given two adjacent prime ideals $P' <\cdot P$, the valuation domain $S = R_P/P'$ obtained localizing R at P and factorizing over P', is archimedean, with a real-valued valuation w, which is discrete if and only if S is a DVR.

Fixed $\lambda > 0$, we have the length function L_w: Mod(S) $\to$ **R*** defined in Example 4:

$$L_w(S/J) = \lambda \cdot \inf \{w(x) : x \in J \leq S\}.$$

From the length function L_w we derive the length function

$$L: \mathrm{Mod}(R) \to \mathbf{R}^*$$

defined as follows.

Given an R-module M,

$$L(M) = \begin{cases} L_w(M_{P/P'}) & \text{if } P'M = 0 \\ & (M_{P/P'} \text{ is an S-module}) \\ \infty & \text{otherwise} \end{cases} \qquad L(R/I) = \begin{cases} 0 & \text{if } P < I \\ L_w(S/IS) & \text{if } P' < I \leq P \\ \infty & \text{if } I \leq P' \end{cases}$$

The connection between $\mathrm{ent}_L : \mathrm{Mod}(R) \to \mathbf{R}^*$ and $\mathrm{ent}_w : \mathrm{Mod}(S) \to \mathbf{R}^*$ is:

$$\varphi : M \to M \text{ induces } \psi : M_P/P'M_P \to M_P/P'M_P \text{ and } \mathrm{ent}_L(\varphi) = \mathrm{ent}_w(\psi).$$

Length functions of this type are the only non-discrete ones, provided R_P/P' is a non-discrete archimedean domain, equivalently, if $P^2 = P$.

If $P' = 0$, then P is the maximal ideal of R and $w = v$ is the valuation of the archimedean valuation domain R. If $\lambda = 1$, $L = L_V$ is the valuation length of Example 4:

for every ideal $I \leq R$ we have: $\quad L_V(R/I) = \inf\{v(a) : a \in I\}$.

3. Valuation entropy ent_V and intrinsic valuation entropy $\mathrm{ent}_V^{\sim}$

From now on, we consider a valuation domain R with valuation

$$v : R \to \Gamma_{\geq 0} \cup \{\infty\}$$

where Γ is a dense subgroup of the real numbers, the associated valuation length

$L_V : \mathrm{Mod}(R) \to \mathbf{R}^*$, and the L_V-entropy (<u>valuation entropy</u>), denoted by ent_V.

Let φ be an endomorphism of M, $H \leq M$ and $L_V(H) < \infty$; the n^{th}–φ-trajectory of H is

$$T_n(\varphi,H) = H + \varphi H + \ldots + \varphi^{n-1}H.$$

It is easy to see that for each n:

$$L_V(T_{n+1}(\varphi,H)/T_n(\varphi,H)) \geq L_V(T_{n+2}(\varphi,H)/T_{n+1}(\varphi,H))$$

from which, using the "ε-technique" (see below), we derive:

$$\mathrm{ent}_V(\varphi,H) = \inf_n L_V(T_{n+1}(\varphi,H)/T_n(\varphi,H)).$$

The valuation entropy $ent_V(\varphi)$ was investigated by Zanardo, who computed it explicitely for endomorphisms of cyclic torsion trajectories:

- if $\varphi : T(\varphi,xR) \to T(\varphi,xR)$ is an endomorphism, where $T(\varphi,xR) = \sum_{n\geq 0} \varphi^n xR$ and xR is torsion, then $ent_V(\varphi) = L_V(R/A)$, where $A = \bigcup_{n>1} Ann(T_n(\varphi,xR)/T_{n-1}(\varphi,xR))$;

it follows that:

- if $\varphi : T(\varphi,F) \to T(\varphi,F)$ is an endomorphism, where F is a finitely generated torsion module, then $ent_V(\varphi) \leq L_V(F)$; in particular, $ent_V(\varphi) < \infty$.

Using these results we proved a kind of Uniqueness Theorem for ent_V, the same result being still unproved for the entropy ent_L, when L is arbitrary non-discrete.

Observe that

$$T_{n+1}(\varphi,H) \,/\, T_n(\varphi,H) \cong (T_{n+1}(\varphi,H)/H) \,/\, (T_n(\varphi,H)/H)$$

so the quantity $\inf_n L_V(T_{n+1}(\varphi,H)/T_n(\varphi,H))$ makes sense even if $L_V(T_n(\varphi,H)/H) < \infty$ for each n; this is ensured just assuming that $L_V(T_2(\varphi,H)/H) = L_V((H+\varphi H)/H) < \infty$.

DEFINITIONS. 1) Let M be an R-module and $\varphi : M \to M$ an endomorphism. A submodule H of M is *φ-inert* if $L_V((H+\varphi H)/H) < \infty$.

2) The *intrinsic valuation entropy* of φ with respect to the φ-inert submodule H is

$$\mathrm{ent}_V\tilde{}(\varphi,H) = \lim_{n\to\infty} L_V(T_n(\varphi,H)/H)/n \,.$$

3) The *intrinsic valuation entropy* of φ is: $\mathrm{ent}_V\tilde{}(\varphi) = \sup\{ \mathrm{ent}_V\tilde{}(\varphi,H) : H \ \varphi\text{-inert} \}$.

PROPOSITION. $ent_V\tilde{}(\varphi,H) = \inf_n L_V(T_{n+1}(\varphi,H)/T_n(\varphi,H))$.

<u>Proof with the "ε-technique"</u>

Let $T_n = T_n(\varphi,H)$ for all n. We know that $L_V(T_{n+1}/T_n) \geq L_V(T_{n+2}/T_{n+1})$ for all n.

Let $\alpha = \inf_n L_V(T_{n+1}/T_n)$ and fix $\varepsilon > 0$. Then

$$L_V(T_{n+1}/T_n) = L_V((T_{n+1}/H)/(T_n/H)) < \alpha + \varepsilon$$

for $n \geq h$ large enough. By induction on k, we get

$$L_V((T_{h+k}/H) = L_V(T_h/H)) + \sum_{0\leq i\leq k-1} L_V((T_{h+i+1}/H)/(T_{h+i}/H))$$

therefore we deduce

$$L_V(T_h/H)) + k\alpha \leq L_V((T_{h+k}/H) \leq L_V(T_h/H)) + k(\alpha + \varepsilon).$$

Diividing by h+k, from the first inequality we obtain $ent_V\tilde{}(\varphi,H) \geq \alpha$.

From the second inequality we obtain $ent_V\tilde{}(\varphi,H) \leq \alpha + \varepsilon$, so we are done.

A submodule of finite valuation length is φ-inert for all φ, hence $\mathrm{ent}_V(\varphi) \leq \mathrm{ent}_V^{\sim}(\varphi)$.

PROPOSITION. If M is a torsion R-module, then $\mathrm{ent}_V(\varphi) = \mathrm{ent}_V^{\sim}(\varphi)$.

Proof with the "ε-technique" If H is φ-inert in M, fixed a positive real number $\varepsilon > 0$, there exists a finitely generated submodule F of H such that

$$L_V((H+\varphi H)/H) - L_V((H+\varphi F)/H) < \varepsilon$$

(because $L(H) = \sup_F L(F)$), hence, for each $n > 1$,

$$L_V(T_n(\varphi,H)/H) - L_V((T_n(\varphi,F)+H)/H) < n\varepsilon$$

therefore $$L_V(T_n(\varphi,H)/H) - L_V(T_n(\varphi,F)/F) < n\varepsilon.$$

Being F finitely generated torsion, $L_V(F) < \infty$, so F is φ-inert, so we derive:

$$\mathrm{ent}_V^{\sim}(\varphi,H) - \mathrm{ent}_V^{\sim}(\varphi,F) < \varepsilon.$$

Now F finitely generated torsion implies $L_V(F) < \infty$, hence $\mathrm{ent}_V^{\sim}(\varphi,F) = \mathrm{ent}_V(\varphi,F)$, therefore $\mathrm{ent}_V^{\sim}(\varphi,H) < \mathrm{ent}_V(\varphi,F) + \varepsilon \leq \mathrm{ent}_V(\varphi) + \varepsilon$. Being ε arbitrary, we are done.

4. A new tool: the anti-trajectories

Anti-trajectories have been introduced for Abelian groups by S-Virili, inspired by a 2015 paper by Willis on the scale function of continuous endomorphisms of totally disconnected locally compact groups.

Let M be an R-module, $\varphi : M \to M$ an endomorphism and K a submodule.

The *n^{th}-anti-trajectory* $A_n(\varphi,K)$ of K is defined by induction on n:

$$A_1(\varphi,K) = K \qquad \text{and}$$

$$A_n(\varphi,K) = K + \varphi^{-1}A_{n-1}(\varphi,K) \qquad \text{for } n > 1.$$

Note that in general : $A_n(\varphi,K) \geq K + \varphi^{-1}K + \varphi^{-2}K + \ldots + \varphi^{-n+1}K.$

The *anti-trajectory* of K is: $A(\varphi,K) = \bigcup_n A_n(\varphi,K)$.

$A(\varphi,K)$ contains $\varphi^{-1}A(\varphi,K)$ and $A(\varphi,K) = K$ if and only if $\varphi^{-1}(K) \leq K$.

The anti-trajectories are related to the trajectories as follows:

$$T_{n+1}(\varphi,K) \,/\, T_n(\varphi,K) \cong A_{n+1}(\varphi,K) \,/\, \varphi^{-1}A_n(\varphi,K)$$

$$T_{n+1}(\varphi,A(\varphi,K)) \,/\, T_n(\varphi,A(\varphi,K)) \cong A(\varphi,K) \,/\, \varphi^{-1}A(\varphi,K) \qquad \text{for all } n.$$

The important facts concerning the anti-trajectories are:

♦ if K is φ-inert in M, then also $A(\varphi,K)$ is φ-inert and $\mathrm{ent}_V^{\sim}(\varphi,K) = \mathrm{ent}_V^{\sim}(\varphi,A(\varphi,K))$

♦♦ $\mathrm{ent}_V^{\sim}(\varphi,A(\varphi,K)) = L_V(A(\varphi,K) \,/\, \varphi^{-1}A(\varphi,K))$.

It follows that if K is φ-inert in M and $\varphi^{-1}(K) \leq K$ (i.e., $A(\varphi,K) = K$), then

$$ent_V^{\sim}(\varphi,K) = L_V(K/\varphi^{-1}K)$$ [limit-free !]

hence: $ent_V^{\sim}(\varphi) = \sup\{ L_V(K/\varphi^{-1}K) : M \geq K \ \varphi\text{-inert and } \varphi^{-1}(K) \leq K \}$.

This "limit-free formula" will be very useful in proving the inequality "≤" in the

ADDITION THEOREM. Let M be an R-module, $\varphi : M \to M$ an endomorphism and N a φ-invariant submodule of M. Then:

$$ent_V^{\sim}(\varphi) = ent_V^{\sim}(\varphi_{\restriction N}) + ent_V^{\sim}(\overline{\varphi})$$

where $\varphi_{\restriction N} : N \to N$ and $\overline{\varphi} : M/N \to M/N$ are the induced endomorphisms.

5. $\text{ent}_V\tilde{}$ is a length function on Mod(R[X])

A module M with an endomorphism $\varphi : M \to M$ is viewed as an R[X]-module, denoted by M_φ, with the multiplication by the scalar X given by the action of φ. A φ-invariant submodule N is an R[X]-submodule and AT simply says that the map

$$\text{ent}_V\tilde{} : \text{Mod}(R[X]) \to \mathbf{R}^*$$

given by $\text{ent}_V\tilde{}(M_\varphi) = \text{ent}_V\tilde{}(\varphi)$, is an additive invariant (it is an invariant since conjugate endomorphisms have equal intrinsic valuation entropy: exercise).

The fact that $\text{ent}_V\tilde{}$ is <u>upper-continuous</u> can be proved using the "ε-technique", thus, if we prove AT, we have that $\text{ent}_V\tilde{} : \text{Mod}(R[X]) \to \mathbf{R}^*$ is a length function.

SKETCH OF THE PROOF OF: $\mathrm{ent}_V\tilde{}(\varphi) \leq \mathrm{ent}_V\tilde{}(\varphi_{IN}) + \mathrm{ent}_V\tilde{}(\varphi^{-})$

Step 1. Let $\mathrm{Ker}_\infty(\varphi) = \bigcup_n \mathrm{Ker}(\varphi^n)$ be the hyperkernel of φ. The induced map

$$\psi : M/\mathrm{Ker}_\infty(\varphi) \to M/\mathrm{Ker}_\infty(\varphi)$$

is injective and using the "limit-free formula" one gets $\mathrm{ent}_V\tilde{}(\varphi) = \mathrm{ent}_V\tilde{}(\psi)$.

Pass to modules over the Laurent polynomials (endomorphisms induced by φ become isomorphisms) tensorizing R[X]-modules by $R[X^{\pm 1}]$.

Step 2. Using Step 1, $\mathrm{ent}_V\tilde{}(M_\varphi) = \mathrm{ent}_V\tilde{}(M_\varphi \otimes_{R[X]} R[X^{\pm 1}])$

Step 3. Proof of the inequality "≤ " of AT for exact sequences in Mod($R[X^{\pm 1}]$), using the fact that endomorphisms are bijections, again using the "limit-free formula".

Step 4. Passage from Mod($R[X^{\pm 1}]$) to Mod(R[X]) using the fact that $R[X^{\pm 1}]$ is a flat R[X]-module, so the exact sequence $0 \to N \to M \to M/N \to 0$ gives rise to the exact sequence

$$0 \to N \otimes_{R[X]} R[X^{\pm 1}] \to M \otimes_{R[X]} R[X^{\pm 1}] \to M/N \otimes_{R[X]} R[X^{\pm 1}] \to 0.$$

Step 5. Step 2 and the inequality "≤ " in Mod($R[X^{\pm 1}]$) give the inequality "≤ " in Mod(R[X]).

SKETCH OF THE PROOF OF: $\mathrm{ent}_V\tilde{}(\varphi) \geq \mathrm{ent}_V\tilde{}(\varphi_{IN}) + \mathrm{ent}_V\tilde{}(\varphi^{-})$.

Once the inequality "≤" is proved, we have 5 steps in order to show that

$$\mathrm{ent}_V\tilde{}(\varphi) \geq \mathrm{ent}_V\tilde{}(\varphi_{IN}) + \mathrm{ent}_V\tilde{}(\varphi^{-}).$$

<u>Step 1</u>. AT holds if M is torsion, since in this case $\mathrm{ent}_V\tilde{}(\varphi) = \mathrm{ent}_V(\varphi)$ and by [SV]

<u>Step 2</u>. Using Upper Continuity (UC) and the "ε-technique", reduce to the case of $M = T(\varphi,F)$ a finitely generated $R[X]$-module (F finitely generated R-module), in which case: $\mathrm{ent}_V\tilde{}(\varphi) < \infty \iff \mathrm{rk}_R(M) < \infty$

<u>Step 3</u>. If M is <u>torsion-free</u> and $\mathrm{rk}_R(M) = n$, then $\mathrm{ent}_V\tilde{}(\varphi) = \mathrm{ent}_V\tilde{}(\varphi,F)$ for $M \geq F$ free of rank n, from which AT easily follows.

Step 4. If M is mixed and $\mathrm{rk}_R(M) = n$, then $\mathrm{ent}_V^{\sim}(\varphi) \geq \mathrm{ent}_V^{\sim}(\varphi|_{t(M)}) + \mathrm{ent}_V^{\sim}(\varphi^*)$

where $\varphi^* : M/t(M) \to M/t(M)$ is the induced endomorphism.

Step 5. Induct on $\mathrm{rk}_R(M) = n$ using the 3x3 diagram of R[X]-modules

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & t(N) & \to & t(M) & \to & (t(M)+N)/N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & M & \to & M/N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N/t(N) & \to & M/t(M) & \to & M/(t(M)+N) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$ent_V^{\sim}$ is additive

- on the first row by Step 1 (the torsion case)
- on the third row by Step 3 (the torsion-free case)
- on the first and second columns by Step 4

- by induction on $rk_R(M)$ one can prove that it is additive on the third column, if $N \leq t(M)$ directly, otherwise $rk_R(M/N) < rk_R(M)$ holds and use induction

- the 3x3 diagram ensures that $ent_V^{\sim}$ is additive on the second row, that is, AT holds.

6. Intrinsic Yuzvinsky Formula and Uniqueness theorem

The Intrinsic Yuzvinski Formula for the intrinsic entropy $\widetilde{\mathrm{ent}}$ in the setting of Abelian groups, proved by Dikranjan-Giordano Bruno- S -Virili in 2015, states:

(IYF) given a linear transformation $\varphi : \mathbf{Q}^n \to \mathbf{Q}^n$, the entropy $\widetilde{\mathrm{ent}}(\varphi)$ coincides with log(s), where s is the minimal common multiple of the denominators of the rational numbers appearing in the (monic) characteristic polynomial $p_\varphi(X)$ of φ over $\mathbf{Q}$.

The number s is the minimal positive integer such that $sp_\varphi(X)$ is a primitive polynomial of $\mathbf{Z}[X]$. In strict analogy with this result, we prove the following:

THEOREM (Intrinsic Yuzvinski Formula) Let R be an archimedean non-discrete valuation domain with valuation v and field of quotients Q. Let $\varphi : Q^n \to Q^n$ be a linear transformation. Then $\mathrm{ent}_V\tilde{}(\varphi) = v(s)$, where $s \in R$ is an element of minimal value such that $sp_\varphi(X) \in R[X]$, with $p_\varphi(X)$ the characteristic polynomial of φ over Q.

So the intrinsic valuation entropy of an endomorphism of a finite dimensional vector space over the field of fractions of R may be immediately derived from its characteristic polynomial.

The proof imitates the proof of the Abelian group setting, making use of the following result of independent interest.

PROPOSITION. Let M be a torsion-free R-module, φ: M → M an endomorphism, $x \in M$ an element generating a φ-trajectory $T(\varphi,xR)$ of finite rank n. Then

(1) $T_n(\varphi, x) = xR \oplus \varphi xR \oplus \varphi^2 xR \oplus \ldots \oplus \varphi^{n-1}xR$ is free of rank n;

(2) there exists a polynomial $f(X) \in R[X]$ of degree n, with content $c(f(X)) = R$ and leading coefficient $s \in R$, such that $f(\varphi)(x) = 0$;

(3) $R[X]/(f(X)) \cong T(\varphi,xR)$;

(4) $T_{k+1}(\varphi, xR)/T_k(\varphi,xR) \cong R/sR$ for every $k \geq n$;

(5) if s is a unit, then $T(\varphi,xR) = T_n(\varphi,xR)$ is free of rank n, otherwise the quotient $T(\varphi,xR)/T_n(\varphi,xR)$ is a uniserial divisible module isomorphic to Q/R.

SKETCH OF THE PROOF OF THE INTRINSIC YUZVINSKI FORMULA (IYF)

$Q[X]$ PID => $Q^n_\varphi = V_\varphi = V_1 \oplus \ldots \oplus V_r$ with the V_i torsion cyclic $Q[X]$-modules

$p_\varphi(X) = \prod_i p_{\varphi i}(X)$, with $p_{\varphi i}(X)$ characteristic polynomial of $\varphi_i = \varphi_{|V_i}$

by AT, it is enough to prove for each V_i, so, wlog, V_φ cyclic $Q[X]$-module =>

$$\Rightarrow V = xQ \oplus \varphi xQ \oplus \ldots \oplus \varphi^{n-1}xQ .$$

Then $F = xR \oplus \varphi xR \oplus \ldots \oplus \varphi^{n-1}xR$ is φ-inert in V and $\mathrm{ent}_V^{\sim}(\varphi) = \mathrm{ent}_V^{\sim}(\varphi,F)$

Now we have: $$\mathrm{ent}_V^{\sim}(\varphi,F) = \inf_k L_V(T_{k+1}(\varphi,F)/T_k(\varphi,F))$$

where $T_k(\varphi,F) = T_{k+n-1}(\varphi,xR)$, so, by (4) in the Proposition:

$$\mathrm{ent}_V^{\sim}(\varphi) = L_V(R/sR) = v(s).$$

The conclusion follows, since $f(X) = sp_\varphi(X)$ if V is a cyclic φ-trajectory.

Using AT and IYF one can prove the following

UNIQUENESS THEOREM .Let R be an archimedean non-discrete valuation domain with valuation $v : R \to \mathbf{R}^*$, and let $L_v : \mathrm{Mod}(R) \to \mathbf{R}^*$ be the induced non-discrete length function. The intrinsic valuation entropy $\mathrm{ent}_v^{\sim}$ is the unique length function L_X on Mod(R[X]) such that:

(1) $L_X(M \otimes_R R[X])) = L_v(M)$ for all modules M (of finite length);

(2) $L_X(V_\varphi) = v(s)$, for any linear transformation $\varphi: V \to V$ of a finite dimensional Q-vector space, where $s \in R$ is an element of minimal value such that $sp_\varphi(X) \in R[X]$, where $p_\varphi(X)$ is the (monic) characteristic polynomial of φ over Q.