Generalized amenability of topological groups and Banach representations

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Plan

• Generalized (extreme) amenability of topological groups: using tame dynamical systems "instead" of the fixed points

Some Examples: $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{H}_+(\mathbb{R}/\mathbb{Z})$, $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}, \circ)$, ...

• Tame dynamical systems: some recent and new results.

• (Recent) ultrahomogeneous actions $G \curvearrowright (X, \circ)$ on circularly ordered sets.

• (New) Group actions on dendrons are tame.

• Questions

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Well known definitions

A topological group G is

1. *extremely amenable* if every continuous action on a compact space admits a fixed point.
   (Equiv.: universal minimal G-system $M(G)$ is trivial).

2. *amenable* if every continuous affine action on a compact convex space admits a fixed point.
   (Equiv.: universal irreducible affine G-system $IA(G)$ is trivial).

An Idea

idea in the context of (extreme) amenability:

"let us free fixed point"

replacing fixed point by a "dynamically small" G-system from some class **P** of compact G-systems.

let us free fixed point

Let **P** be a (nice) class of (dynamically small) compact G-spaces. Assume, at least, $\mathbf{P} \supseteq \{\text{one point trivial } G\text{-systems}\}$.

Definition

We say that a topological group G is:

1. **intrinsically P** if every continuous action of G on a compact space X admits a compact G-subsystem $Y \subseteq X$ such that $(G, Y) \in \mathbf{P}$.
2. **convexly intrinsically P** if every continuous affine action of G on a compact affine space X admits a compact G-subsystem $Y \subseteq X$ such that $(G, Y) \in \mathbf{P}$.

$$\begin{array}{ccc} \text{extreme amenability} & \Longrightarrow & \text{int-}\mathbf{P} \\ \Downarrow & & \Downarrow \\ \text{amenability} & \Longrightarrow & \text{conv-int-}\mathbf{P} \end{array}$$

We study the case $\mathbf{P} = \{\text{Tame DS}\}$.

Definition

(General compact G-system X) $G \times X \to X$ is tame iff
$\forall p \in E(X), f \in C(X),\ f \circ p : X \to \mathbb{R}$ has PCP
(Point of Cont. Prop.) [a]

[a](Envel. sem. $E(X) := \mathrm{cls_p(G)} \subset \mathrm{X^X}$)

$\{\text{Tame}\}$ is closed under products, factors, subsystems.

Metrizable Tame DS

Definition

(for metrizable X) $G \times X \to X$ is tame iff $cardE(X) \leq 2^{\aleph_0}$

Example

$(\mathrm{H}_+[0,1],[0,1])$ is tame.

$$G = \mathrm{H}_+[0,1] \hookrightarrow E(\mathrm{H}_+,[0,1]) \subset \textit{Helly compact}$$

which is first countable (and cardinality $= 2^{\omega}$).

Some interesting classes of tame actions

1. $\underbrace{\text{WAP} \subset \text{HNS} \subset \text{Tame}}_{\text{dynamically "small" systems}} \subset$ Dynamical systems

   (smallness via Dynamical version of BFT dichotomy for compact metric DS)
2. (Ellis, Akin) Projective actions $(GL_n(\mathbb{R}), \mathbb{P}^{n-1})$ and $(GL_n(\mathbb{R}), \mathbb{S}^{n-1})$
3. circularly (e.g., linearly) ordered dynamical systems
4. Sturmian like $\mathbb{Z}$-subshifts $X \subset \{0,1\}^{\mathbb{Z}}$ in symbolic dynamics
5. almost canonical model sets (tilings, models of quasicrystals) [Aujogue15], [Aujogue-Kellendonk15], ...
6. (New) Continuous group actions on (local) dendrons D
   Corollary: $(Homeo(D), D)$ is Rosenthal representable

Tame dynamical systems – more details

Definition (Köhler95)

$f \in C(X)$ is said to be *regular* (tame, in the terminology of Glasner) if the family of functions fG does not contain an independent sequence. Notation: $f \in \mathrm{Tame}(X)$.
(G, X) is tame if $\mathrm{Tame}(X) = C(X)$.

Independent sequences of functions

Definition

A sequence $\{f_n : X \to \mathbb{R}\}_{n\in\mathbb{N}}$ of functions on a set X is **independent** if $\exists\ a < b$ s.t.

$$\bigcap_{n\in P} f_n^{-1}(-\infty, a) \cap \bigcap_{n\in M} f_n^{-1}(b, \infty) \neq \emptyset$$

for all finite disjoint subsets P, M of $\mathbb{N}$.

Example

The sequence of projections on the Cantor set $\{\pi_m : \{0,1\}^{\mathbb{Z}} \to \{0,1\}\}_{m\in\mathbb{Z}}$ is independent (Pointwise closure of this family is $\beta\mathbb{Z}$.

Example (why Bernoulli shift system $(\mathbb{Z}, \{0,1\}^{\mathbb{Z}})$ is not tame:)

projection $\pi_0 : \{0,1\}^{\mathbb{Z}} \to \{0,1\}$ is not a tame function (because $\pi_0 G = \{\pi_k : k \in \mathbb{Z}\}$ is independent).

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Remark (Gl-Me-06, Gl-Me-Usp-08)

If X is metrizable then TFAE:

1. compact G-system X is tame.
2. card $E(X) \leq 2^{\omega}$.
3. In $E(X)$ the topology can be defined by conv. sequences.
4. $E(X)$ does not contain a topological copy of $\beta(\mathbb{N})$.
5. (G, X) is representable on a (separable) Rosenthal Banach space.

Representations of actions on Banach spaces

Which actions $G \times X \to X$
(X is not necessarily compact)
are representable on Banach spaces $V \in \mathcal{K}$?

$$\begin{array}{ccc} G \times X & \longrightarrow & X \\ {\scriptstyle h}\downarrow \quad \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ Is(V) \times V^* & \longrightarrow & V^* \end{array}$$

h is a contin. homomorphism, α is weak-star continuous bounded

Why just tame DS ?

Which continuous actions of G on (not necessarily, compact) X are representable on "small" Banach spaces ?

$$\underbrace{\text{WAP} \subset \text{HNS} \subset \text{Tame}}_{\text{dynamically "small" systems}} \subset \text{Dynamical systems}$$

$$\underbrace{Refl. \subset Asplund \subset Rosenthal}_{\text{"small" Banach spaces}} \subset \text{Banach spaces}$$

♡ role of tame systems in "Dynamical BFT dichotomy"

♡ role of Rosenthal Ban. spaces in Rosenthal l_1-dichotomy

Remark

a separable Banach space V is Rosenthal iff $l_1 \not\subseteq V$ iff $card(V^{**}) = card(V) = 2^{\omega}$

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nothing new if $\mathbf{P} =$ WAP (or, even, if $\mathbf{P} =$HNS)

Equic $=$ AP $\subset$ WAP $\subset$ HNS $\subset$ Tame $\subset$ DS

Lemma ("collapsing effect")

{*convexly intrinsically* **Equic**} $=$ {*convexly intrinsically* **WAP**} $=$ {*convexly intrinsically* **HNS**} $=$ *usual amenability*

Proof.

Every minimal HNS (e.g., WAP) G-system is equicontinuous (AP), hence distal. Then it admits invariant probability measure (by Furstenberg's fixed point theorem). □

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circular analog of Pestov's theorem

Remark

Every **linearly** ordered compact minimal G-space is trivial.

[In contrast to **circular** orders !]

Theorem

Let $G \leq \mathrm{Aut}(X_\circ)$ act ultra-homogeneously on a circularly ordered set $X_\circ$. Then (G, τ_p) is intrinsically c-ordered (i.e., $M(G)$ is a circularly ordered G-system)

Theorem

Furthermore, if X is countable then $M(G) = M(\mathrm{Aut}(\mathbb{Q}_\circ))$ is metrizable and every continuous action of G on a compact space admits a circularly ordered compact G-subspace from this list: $(M(\mathrm{Aut}(\mathbb{Q}_\circ)), \mathbb{T},$ or the fixed point).

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Circular (cyclic) order

Definition ([Huntington], [Cech], [Kok], ...)

Circular order on a set X is a ternary relation $R \subset X^3$ on X s.t.

1. Cyclicity: $[a, b, c] \Rightarrow [b, c, a]$;
2. Asymmetry: $[a, b, c] \Rightarrow (a, c, b) \notin R$;
3. Transitivity: $\begin{cases} [a, b, c] \\ [a, c, d] \end{cases} \Rightarrow [a, b, d]$;
4. Totality: if $a, b, c \in X$ are distinct, then $[a, b, c]$ or $[a, c, b]$.

Constructions for circular orders

- inverse limit
- "Split construction" $\mathrm{Split}(X_\circ; A)$ – doubling elements $A \subset X_\circ$
- *lexicographic product* $X_\circ \times L_<$

of a c-ordered $X_\circ$ and a linearly ordered $L_<$.

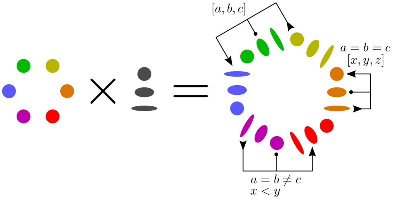


Figure: c-ordered lexicographic product (from Wikipedia)

Sturmian like system $X_\alpha \subset \mathbb{T}_\mathbb{T} := \mathbb{T} \times \{0, 1\}$ "double circle"

Circularly ordered dynamical systems

Definition

We say that a compact G-system (X, τ) is circularly orderable if there exists a τ-compatible circular order R on X such that X is COTS and every g-translation $\tilde{g} : X \to X$ is C-OP.
Denote by CODS the class of all c-orderable dynamical systems.

For every linearly (circularly) ordered **compact** space X and every topological subgroup $G \leq H_+(X)$, with its compact open topology, the corresponding action $G \curvearrowright X$ defines a linearly (circularly) ordered G-system.

Theorem

$\mathrm{LODS} \subset \mathrm{CODS} \subset \{\textit{Rosenthal representable}\} \subset \{\textit{Tame}\}$

Sturmian systems are circularly ordered

Example

Sturmian like symbolic system $X_\alpha \subset \{0,1\}^{\mathbb{Z}}$ (rotation by angle α) is a **circularly ordered** $\mathbb{Z}$-system embedded into the c-ordered lexicographic order $\mathbb{T}_{\mathbb{T}} := \mathbb{T} \times \{-,+\}$
(split any point of the dense orbit of 0 on $\mathbb{T}$).

Example

Moreover, the enveloping semigroup

$$E(X_\alpha) = \mathbb{T}_{\mathbb{T}} \cup \mathbb{Z} \qquad \subset \mathbb{T} \times \{-,0,+\} \text{ (lexic. prod.)}$$

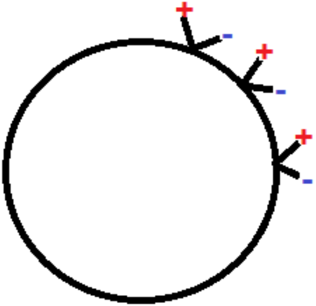
is also a **circularly ordered** system.

Remark

Such "animals" are not metrizable but they are "tamed animals"

Sturmian system

split the points of the orbit $\langle \mathrm{Rot}(\alpha) \rangle$ in T



circular analog of Pestov's theorem

[Gl-Me, arxiv March 2018]

Theorem

Let a subgroup $G \leq \mathrm{Aut}(X_\circ)$ act ultra-homogeneously on a circularly ordered set $X_\circ$. Then (G, τ_p) is intrinsically c-ordered, i.e., $M(G)$ is a c-ordered G-system.

Theorem

Furthermore, if X is countable then $M(G) = M(\mathbb{Q}_\circ)$ is metrizable and every continuous action of G on a compact space admits a circularly ordered compact G-subspace ($M(\mathbb{Q}_\circ)$, $\mathbb{T}$, or the fixed point).

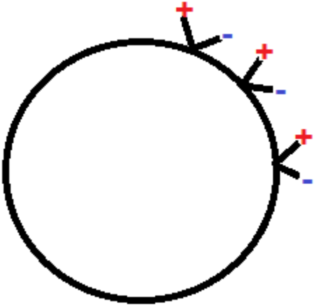
Corollary

The following topological groups are intrinsically circularly ordered

1. *Polish group* $G := \mathrm{Aut}(\mathbb{Q}_\circ)$*;*
   Furthermore, $M(G)$ *is metrizable.*
2. $H_+(\mathbb{T})$ *in the pointwise topology with respect to the action* $H_+(\mathbb{T}) \curvearrowright (\mathbb{T}, \tau_{discr})$*;*
3. *Polish group* $H_+(\mathbb{T})$ *in the compact open topology;*
   ($2 \Rightarrow 3$*:* $H_+(\mathbb{T}, \tau_{discr}) \to H_+(\mathbb{T})$ *is a cont. dense homom.)*
4. **Thompson's circular group** T *with the pointwise topology (acting ultrahomogeneously on "circled dyadic rationals"* $D_\circ$*).*

$M(Aut(Q_o))$ universal minimal space

split the points of Q_o in T



Automatic continuity and **metr-int-c-ord**

Corollary

For every action of the **discrete** *group* $G := \mathrm{Aut}(\mathbb{Q}_\circ)$ *by homeomorphisms on a metric compact space there exists a compact circularly ordered* G*-subsystem.*

Lemma

The Polish group $\mathrm{Aut}(\mathbb{Q}_\circ)$ *has the automatic continuity property (every group homomorphism* $h : \mathrm{Aut}(\mathbb{Q}_\circ) \to H$ *to a separable topological group* H *is continuous).*

Proof.

by Rosendal-Solecki thm Polish group $\mathrm{Aut}(\mathbb{Q}_<)$ has the automatic continuity property. $\mathrm{Aut}(\mathbb{Q}_<)$ open subgroup of $\mathrm{Aut}(\mathbb{Q}_\circ)$. □

Some new classes of Tame DS

[Gl-Me, arxiv June 2018]

- Group actions on (local) dendrons.
- Monotone actions on compact median algebras.

Hint: $median(u, v, w) = [u, v] \cap [u, w] \cap [v, w]$

Theorem

Let D be a dendron. $\forall$ topol. gr. action $G \curvearrowright D$, the dynamical G-system D is Rosenthal representable, hence also tame.

Sketch: $CM(D)$ is a G-invariant point separating family which **has no independent sequence**.

$\forall \ \phi \in cl_p(CM(D))$ is fragmentable that is, has PCP.

Lemma

Let $\phi : D \to \mathbb{R}$ be a monotone map (preserves the betweenness). Then ϕ has PCP (Baire 1 if D is a dendrite).

Theorem

For every compact median pretree X and its automorphism group $G = H_+(X, R)$ the action of the topological group G on X is Rosenthal representable.

Corollary

Let X be a $\mathbb{Z}$-tree. Denote by $Ends(X)$ the set of all its ends. Then for every monotone group action $G \curvearrowright X$ with continuous transformations the induced action of G on the compact space $\widehat{X} := X \cup Ends(X)$ is Rosenthal representable.

Example

Polish group $G := \mathrm{Homeo}_+(\mathbb{T})$ is int-tame and nonamenable.

Sketch: $M(G) = \mathbb{T}$ (by [Pestov98]).
Circular-order preserving dynamical systems are tame [Gl-Me17].

For $G := \mathrm{Homeo}_+(\mathbb{T})$ in any compact G-space we can find a G-circle or a fixed point (which are tame)

Example

$G :=\mathrm{SL}_n(\mathbb{R})$ ($\forall n \geq 2$) is nonamenable but conv-int-tame and not int-tame.

Sketch: for $n = 2$. $IA(G) = P(K)$ – probablity measures on K, where K is the 1-dimensional real projective space $\simeq \mathbb{T} =$ circle. Into any compact **affine** G-space there exists:
1-dim real projective G-space or a fixed point (which are tame).

Some questions

Question

(Besides $SL_n(\mathbb{R})$) Find more natural locally compact groups G which are conv-int-tame but nonamenable.

Question

What if G is DISCRETE ? [a]

Is it true that there exists a nonamenable but conv-int-tame DISCRETE group ?

[a] F_2 does not work here

discrete group G is int-tame iff G is finite

very probably a lc group G is int-tame iff G is compact.

Some questions

Question

Let D be a dendron. When $G = \mathrm{Homeo}(D)$ is minimal ?

YES, if D is a connected linearly ordered space such that $H_+[a, b]$ is not trivial for all $a < b$ (Me-Polev 2016)

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GRAZIE MILLE !