Amenability, Cellular Automata, and Group Graded Rings

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Reinhold Baer's question: which groups have Noetherian group rings? For G polycyclic-by-finite then $\mathbb{Z}G$ is Noetherian. But does the converse hold?

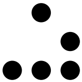

Figure: Conway's Life Game

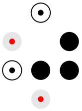

Figure: Conway's Life Game

Based on the group $G = \mathbb{Z}^2$ with transition rules obtaining information through the 9 generators $\{(a, b);\ a, b \in \{1, 0, -1\}\}$.

Transition is a function $[G, \mathbb{F}_2] \to [G, \mathbb{F}_2]$ determined by finitely many rules. That is it is a function from a coinduced module to another coinduced module.

A finite pattern corresponds to an element of the group ring $\mathbb{F}_2 G$. Then the transition restricts to a function $\mathbb{F}_2 G \to \mathbb{F}_2 G$.

Bartholdi's Theorem (2016)
If G is a non-amenable group then there exists a cellular automaton carried by G that admits Gardens of Eden but no mutually erasable patterns.

Myhill's Theorem (1963) [in its general form following additional work of Moore (1962), Machì–Mignosi (1993) and Ceccherini-Silberstein–Machì–Scarabotti (1999).]
If G is amenable then a cellular automaton based on G admits Gardens of Eden if and only if it admits mutually erasable patterns.

Bartholdi's Theorem (2016)
If G is a non-amenable group then there exists a cellular automaton carried by G that admits Gardens of Eden but no mutually erasable patterns.

Kielak noticed that Bartholdi's proof has the following application: If G is non-amenable and K is a field then there exist $m > n$ and an injective KG-module homomorphism $KG^m \to KG^n$.

Group Graded Rings

A ring R is G-graded if

$$R = \oplus_{g \in G} R_g$$

and for each g, h

$$R_g R_g \subseteq R_{gh}.$$

R is strongly G-graded if, in addition,

$$R_g R_g = R_{gh}$$

Ranks

Two fundamental aspects of finite dimensional vector spaces over a field K.

1. If K^n can be spanned by m elements then $m \geq n$.
2. If K^n contains a set of m linearly independent vectors then $m \leq n$.

For a ring R and left R-modules:

1. R has the rank condition if surjections $R^m \to R^n$ only occur when $m \geq n$.
2. R has the *strong rank condition* if injections $R^m \to R^n$ only occur when $m \leq n$.

the rank condition is left right symmetric

$$\begin{array}{c}\text{left strong rank condition}\\ \Downarrow \\ \text{rank condition} \\ \Uparrow \\ \text{right strong rank condition}\end{array}$$

Proof of $\Downarrow$. If $R^m \to R^n$ is surjective then there is a splitting. So that is an inclusion $R^n \hookrightarrow R^m$.

If R fails the strong rank condition then for large enough m, R^m contains a free R-module of infinite rank.

In particular, R has infinite uniform dimension.

If R is a domain then the following are equivalent:

1. R has the strong rank condition
2. R has finite uniform dimension
3. R is an Ore domain
4. R has no right ideal isomorphic to R^2.

Let G be a group and let R be a G-graded ring.

If R is strongly G-graded and R_1 is a domain then the following are equivalent.

1. R has the strong rank condition.
2. R_1 is an Ore domain and G is amenable

If R_g contains a non-zero divisor for each g and R has the strong rank condition then G is amenable.

EXAMPLES

$$\mathbb{C} = \mathbb{R}[i]$$

is $\mathbb{Z}/2\mathbb{Z}$-graded.

$$\mathbb{Q}\left[e^{2\pi i/p}, e^{2\pi i/p^2}, e^{2\pi i/p^3}, e^{2\pi i/p^4}, \ldots\right]$$

is $\mathbb{Z}[1/p]/\mathbb{Z}$-graded.

If R is Noetherian strongly G-graded and R_1 is a domain then G is amenable.

If $\mathbb{Z}G$ is Noetherian then G is an amenable group with max on subgroups.

THE END