Feebly compact semitopological symmetric inverse semigroups of a bounded finite rank

Oleg Gutik

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Let $\lambda$ be an arbitrary non-zero cardinal. A map $\alpha$ from a subset $D$ of $\lambda$ into $\lambda$ is called a *partial transformation* of $\lambda$. In this case the set $D$ is called the *domain* of $\alpha$ and it is denoted by $\text{dom } \alpha$. The image of an element $x \in \text{dom } \alpha$ under $\alpha$ we shall denote by $x \alpha$. Also, the set $\{x \in \lambda : y \alpha = x \text{ for some } y \in Y\}$ is called the *range* of $\alpha$ and is denoted by $\text{ran } \alpha$. The cardinality of $\text{ran } \alpha$ is called the *rank* of $\alpha$ and denoted by $\text{rank } \alpha$. For convenience we denote by $\emptyset$ the empty transformation, that is a partial mapping with $\text{dom } \emptyset = \text{ran } \emptyset = \emptyset$. 
Let $\mathcal{I}_\lambda$ be the set of all partial one-to-one transformations of $\lambda$ together with the following semigroup operation:

$$x(\alpha \beta) = (x\alpha)\beta$$ if $x \in \text{dom}(\alpha \beta) = \{y \in \text{dom} \alpha : y\alpha \in \text{dom} \beta\}$, for $\alpha, \beta \in \mathcal{I}_\lambda$.

The semigroup $\mathcal{I}_\lambda$ is called the symmetric inverse semigroup over the cardinal $\lambda$. 
**Definition (Wagner–Clifford, 1952-1954)**

A semigroup $S$ is called *inverse* if for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such element $y$ is said to be inverse of $x$ and denote by $x^{-1}$. If $S$ is an inverse semigroup, then the map $S \to S$: $x \mapsto x^{-1}$ is called *inversion*.

**Definition**

Put $\mathcal{I}_\lambda^n = \{\alpha \in \mathcal{I}_\lambda : \text{rank } \alpha \leq n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathcal{I}_\lambda^n$ is an inverse subsemigroup of $\mathcal{I}_\lambda$, and moreover $\mathcal{I}_\lambda^n$ is an ideal of $\mathcal{I}_\lambda$, for each $n = 1, 2, 3, \ldots$. The semigroup $\mathcal{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank $\leq n$*. 
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Definitions

A *semitopological* (*topological*) *semigroup* is a Hausdorff topological space with separately continuous (continuous) semigroup operations. Inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology $\tau$ on a semigroup $S$ is defined to be

- *shift-continuous* if for every $a \in S$ the left and right shifts $l_a : S \to S$, $l_a : x \mapsto ax$, and $r_a : S \to S$, $r_a : x \mapsto xa$, are continuous.
- *semigroup* if the semigroup operation in $(S, \tau)$ is continuous;
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\end{itemize}
Generalizations of compactness

- compact
- \( \omega \)-bounded
- \( \omega \)-pracompact
- sequentially compact
- totally countably compact
- countably compact
- total. countably pracompact
- sequentially pracompact
- countably pracompact
- pracompact
- infra H-closed
- \( d \)-feebly compact
- \( \omega \)-bounded
- select. sequent. feebly compact
- selectively feebly compact
- feebly compact
- \( D \)-compact
- \( R \)-compact
- infra H-closed
- pseudocompact

A topological space \( X \) is said to be **compact** if each open cover of \( X \) has a finite subcover.
Generalizations of compactness

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A topological space \(X\) is said to be \(\omega\)-bounded if each countable subset of \(X\) has the compact closure.
A topological space $X$ is said to be $\omega$-pracompact if $X$ contains a dense subset $D$ such that each countable subset of $D$ has the compact closure in $X$ [G.-Ravsky, 2018].
### Generalizations of compactness

- **compact**
- **ω-bounded**
- **ω-pracompact**
- **sequentially compact**
- **totally countably compact**
- **countably compact**
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- **feebly ω-bounded**
- **sequentially feebly compact**
- **feebly compact**
- **d-feebly compact**
- **Ωω-compact**
- **R-compact**
- **infra H-closed**
- **pseudocompact**

A topological space $X$ is said to be **sequentially compact** if each sequence $\{x_i\}_{i \in \mathbb{N}}$ of $X$ has a convergent subsequence in $X$. 
Generalizations of compactness

- compact
- $\omega$-bounded
- $\omega$-pracompact
- sequentially compact
- totally countably compact
- countably compact
- total. countably pracompact
- sequentially pracompact
- countably pracompact
- H-closed
- select. sequent. feebly compact
- selectively feebly compact
- feebly $\omega$-bounded
- sequentially feebly compact
- feebly compact
- $d$-feebly compact
- $\mathcal{D}_\omega$-compact
- $\mathbb{R}$-compact
- infra H-closed
- pseudocompact

A topological space $X$ is said to be **totally countably compact** if each sequence of $X$ contains a subsequence with the compact closure.
A topological space $X$ is said to be *countably compact* if each open countable cover of $X$ has a finite subcover.
A topological space $X$ is said to be *totally countably pracompact* if there exists a dense subset $D$ of $X$ such that each sequence of points of $D$ has a subsequence with the compact closure in $X$ [G.-Ravsky, 2018].
A topological space $X$ is said to be **sequentially pracompact** if there exists a dense subset $D$ of $X$ such that each sequence of points of $D$ has a convergent subsequence [G.-Ravsky, 2018].
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A topological space $X$ is said to be **countably pracompact** if there exists a dense subset $A$ in $X$ such that $X$ is countably compact at $A$ [Arkhanegelskii, 1988].
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A topological space $X$ is said to be $H$-closed if $X$ is a closed subspace of every Hausdorff topological space in which it contained.
A topological space $X$ is said to be \textit{selectively sequentially feebly compact} if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of $X$, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence [Dorantes, Aldama, Shakhmatov, 2017].
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| $\mathcal{O}_\omega$-compact | $\mathcal{R}$-compact | infra H-closed              | pseudocompact

A topological space $X$ is said to be **selectively feebly compact** if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of $X$, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood $W$ of $x$ [Dow, Porter, Stephenson, Woods, 2004].
A topological space $X$ is said to be \textit{feebly $\omega$-bounded} if for each sequence $\{U_n\}_{n \in \mathbb{N}}$ of non-empty open subsets of $X$ there is a compact subset $K$ of $X$ such that $K \cap U_n \neq \emptyset$ for each $n$ [G.-Ravsky, 2018].
A topological space $X$ is said to be \textit{sequentially feebly compact} if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of $X$, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \emptyset\}$ is finite for every open neighborhood $W$ of $x$ [Dow, Porter, Stephenson, Woods, 2004].
Generalizations of compactness

A topological space $X$ is said to be **feebly compact** if each locally finite open cover of $X$ is finite [Bagley, Connell, McKnight, Jr., 1958].
A topological space $X$ is said to be $d$-feebly compact (or DFCC) if every discrete family of open subsets in $X$ is finite.
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A topological space $X$ is said to be *pseudocompact* if $X$ is Tychonoff and each continuous real-valued function on $X$ is bounded.
A topological space $X$ is said to be **infra $H$-closed** provided that any continuous image of $X$ into any first countable Hausdorff space is closed [Hajek, Todd, 1975].
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A topological space $X$ is said to be $Y$-compact for some topological space $Y$, if $f(X)$ is compact, for any continuous map $f: X \to Y$. 
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Metrizable spaces
Old Results

Theorem (G-Pavlyk-Reiter, 2009)

For infinite cardinal $\lambda$ the semigroup $\mathcal{I}_\lambda^1$ does not embed into a Hausdorff countably compact topological semigroup.

Theorem (G-Pavlyk, 2005)

For any infinite cardinal $\lambda$ there exists a unique shift-continuous Hausdorff compact topology on $\mathcal{I}_\lambda^1$.

This topology is the Alexandroff one-point compactification of the discrete space of cardinality $\lambda$ and it will denote by $\tau_{Ac}$.

Theorem (G-Pavlyk, 2005)

Let $\lambda$ by any infinite cardinal. Then every shift-continuous $T_1$-topology on $\mathcal{I}_\lambda^1$ is collectionwise normal and the following statement are equivalent:

(i) $(\mathcal{I}_\lambda^1, \tau)$ is a compact Hausdorff semitopological semigroup;

(ii) $\tau = \tau_{Ac}$;

(iii) $(\mathcal{I}_\lambda^1, \tau)$ is a feebly compact Hausdorff semitopological semigroup.
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(ii) $\tau = \tau_{Ac}$;
(iii) $(I^1_\lambda, \tau)$ is a feebly compact Hausdorff semitopological semigroup.
\((\mathcal{I}_{\lambda}^1, \tau)\) is a Hausdorff semitopological semigroup.
Theorem (G-Lawson-Repovš, 2009)
Let $\lambda$ be any infinite cardinal and $n$ be any positive integer. If a Hausdorff semitopological semigroup $S$ with continuous inversion contains $I_\lambda^n$, then $I_\lambda^n$ is a closed subsemigroup of $S$.

Theorem (G-Reiter, 2010)
Let $\lambda$ be any infinite cardinal, $n$ be any positive integer and $h : I_\lambda^n \to S$ be homomorphism into a Hausdorff semitopological semigroup $S$ with continuous inversion. Then $(I_\lambda^n)h$ is a closed subsemigroup of $S$.

Definition
Let $\mathcal{I}$ be a class of Hausdorff semitopological semigroups. A semigroup $S \in \mathcal{I}$ is called H-closed in $\mathcal{I}$, if $S$ is a closed subsemigroup of any topological semigroup $T \in \mathcal{I}$ which contains $S$ both as a subsemigroup and as a topological space.

Theorem (G-2014)
Let $\lambda$ by any infinite cardinal. Then a semitopological semigroup $(I_\lambda^1, \tau)$ is H-closed in the class of Hausdorff semitopological semigroups if and only if $(I_\lambda^1, \tau)$ is compact.
Old Results

Theorem (G-Lawson-Repovš, 2009)

Let $\lambda$ be any infinite cardinal and $n$ be any positive integer. If a Hausdorff semitopological semigroup $S$ with continuous inversion contains $\mathcal{I}_\lambda^n$, then $\mathcal{I}_\lambda^n$ is a closed subsemigroup of $S$.

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Definition

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Theorem (G-2014)

Let $\lambda$ be any infinite cardinal. Then a semitopological semigroup $(\mathcal{I}_\lambda^1, \tau)$ is H-closed in the class of Hausdorff semitopological semigroups if and only if $(\mathcal{I}_\lambda^1, \tau)$ is compact.
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**Definition**

Let $\mathcal{I}$ be a class of Hausdorff semitopological semigroups. A semigroup $S \in \mathcal{I}$ is called **H-closed** in $\mathcal{I}$, if $S$ is a closed subsemigroup of any topological semigroup $T \in \mathcal{I}$ which contains $S$ both as a subsemigroup and as a topological space.

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**Old Results**

**Theorem (G-Lawson-Repovš, 2009)**

Let $\lambda$ be any infinite cardinal and $n$ be any positive integer. If a Hausdorff semitopological semigroup $S$ with continuous inversion contains $I^n_\lambda$, then $I^n_\lambda$ is a closed subsemigroup of $S$.

**Theorem (G-Reiter, 2010)**

Let $\lambda$ be any infinite cardinal, $n$ be any positive integer and $h : I^n_\lambda \to S$ be homomorphism into a Hausdorff semitopological semigroup $S$ with continuous inversion. Then $(I^n_\lambda)h$ is a closed subsemigroup of $S$.

**Definition**

Let $\mathcal{I}$ be a class of Hausdorff semitopological semigroups. A semigroup $S \in \mathcal{I}$ is called **$H$-closed** in $\mathcal{I}$, if $S$ is a closed subsemigroup of any topological semigroup $T \in \mathcal{I}$ which contains $S$ both as a subsemigroup and as a topological space.

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Let $\lambda$ be any infinite cardinal. Then a semitopological semigroup $(I^1_\lambda, \tau)$ is $H$-closed in the class of Hausdorff semitopological semigroups if and only if $(I^1_\lambda, \tau)$ is compact.
On a $T_1$-semitopological semigroup $(\mathcal{I}_\lambda^1, \tau)$

Any above condition is equivalent to the following: $(\mathcal{I}_\lambda^1, \tau)$ is H-closed in the class of Hausdorff semitopological semigroups.
On the semigroup \((\mathcal{I}_\lambda^n, \tau), n > 1\)

**Question**

Do the above statements hold for a semitopological semigroup \((\mathcal{I}_\lambda^n, \tau)\) for \(n > 1\)?

**Definition (Wagner, 1952)**

On an inverse semigroup \(S\) we define the *natural partial order* \(\preceq\) on \(S\) in the following way:

\[
x \preceq y \quad \text{if and only if there exists an idempotent } e \in S \text{ such that } x = ey.
\]

**Example (G-Reiter, 2010)**

Fix an arbitrary positive integer \(n\). The following family

\[
\mathcal{B}_c = \left\{ U_\alpha(\alpha_1, \ldots, \alpha_k) = \uparrow_{\preceq} \alpha \setminus (\uparrow_{\preceq} \alpha_1 \cup \cdots \cup \uparrow_{\preceq} \alpha_k) \preceq : \right. \\
\left. \alpha_i \in \uparrow_{\preceq} \alpha \setminus \{ \alpha \}, \alpha, \alpha_i \in \mathcal{I}_\lambda^n, i = 1, \ldots, k \right\}
\]

determines a base of the topology \(\tau_c\) on \(\mathcal{I}_\lambda^n\). Then \((\mathcal{I}_\lambda^n, \tau_c)\) is Hausdorff compact semitopological semigroup with continuous inversion.
On the semigroup $(I^n_\lambda, \tau)$, $n > 1$

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Theorem

Let $n$ be an arbitrary positive integer, $\lambda$ be any infinite cardinal and $\tau$ be a $T_1$-shift continuous topology on the semigroup $\mathcal{I}^n_\lambda$. Then the following conditions are equivalent:

(i) $\tau$ is compact;
(ii) $\tau = \tau_c$;
(iii) $\tau$ is countably compact;
(iv) $\tau$ is sequentially compact;
(v) $\tau$ is $\omega$-pracompact;
(vi) $\tau$ is feebly $\omega$-bounded.
On the semigroup \((\mathcal{I}^n_\lambda, \tau),\ n > 1\)

**Theorem**

Let \(n\) be an arbitrary positive integer and \(\lambda\) be an arbitrary infinite cardinal. Then for every shift-continuous \(T_1\)-topology \(\tau\) on the semigroup \(\mathcal{I}^n_\lambda\) the following conditions are equivalent:

- (i) \(\tau\) is sequentially precompact;
- (ii) \(\tau\) is total countably precompact;
- (iii) \(\tau\) is \(d\)-feebly compact;
- (iv) \((\mathcal{I}^n_\lambda, \tau)\) is \(\mathcal{D}_\omega\)-compact.

**Example**

For any infinite cardinal \(\lambda\) and any positive integer \(n \geq 2\) there exists a Hausdorff feebly compact topology \(\tau\) on the semigroup \(\mathcal{I}^n_\lambda\) such that \((\mathcal{I}^n_\lambda, \tau)\) is a non-compact non-semiregular semitopological semigroup.

**Theorem**

Let \(n\) be an arbitrary positive integer and \(\lambda\) be an arbitrary infinite cardinal. Then every shift-continuous semiregular feebly compact \(T_1\)-topology \(\tau\) on \(\mathcal{I}^n_\lambda\) is compact.
On the semigroup \((\mathcal{I}_\lambda^n, \tau), \, n > 1\)

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On a $T_1$-semitopological semigroup $(X^n, \tau)$, $n > 1$
On a semiregular $T_1$-semitopological semigroup $(\mathcal{F}^n, \tau), \ n > 1$
Thank You

Thank You for attention!