Feebly compact semitopological symmetric inverse semigroups of a bounded finite rank

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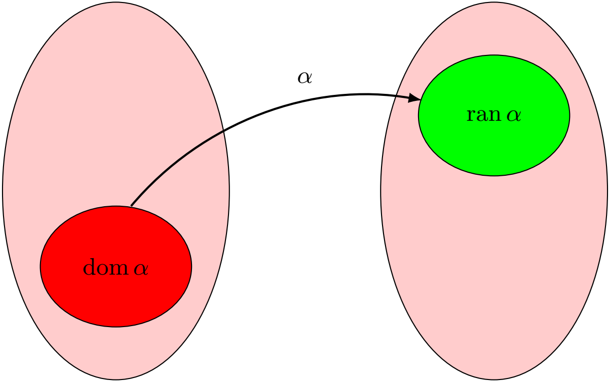
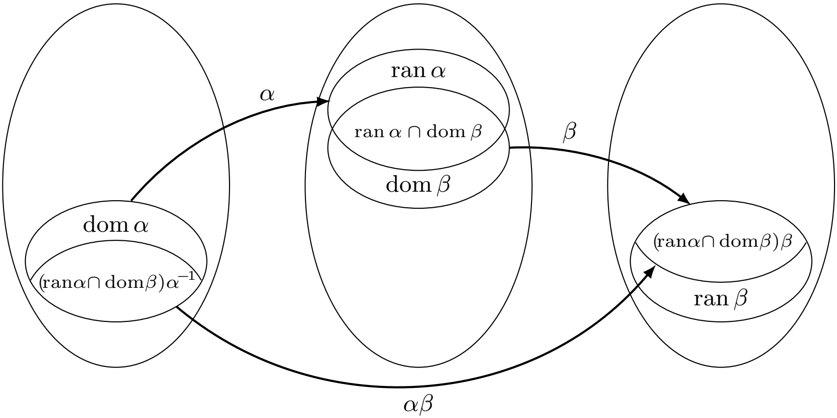
Partial maps



Let λ be an arbitrary non-zero cardinal. A map α from a subset D of λ into λ is called a *partial transformation* of λ. In this case the set D is called the *domain* of α and it is denoted by $\operatorname{dom}\alpha$. The image of an element $x \in \operatorname{dom}\alpha$ under α we shall denote by $x\alpha$ Also, the set $\{x \in \lambda\colon y\alpha = x \text{ for some } y \in Y\}$ is called the *range* of α and is denoted by $\operatorname{ran}\alpha$. The cardinality of $\operatorname{ran}\alpha$ is called the *rank* of α and denoted by $\operatorname{rank}\alpha$. For convenience we denote by $\varnothing$ the empty transformation, that is a partial mapping with $\operatorname{dom}\varnothing = \operatorname{ran}\varnothing = \varnothing$.



Let $\mathscr{I}_\lambda$ be the set of all partial one-to-one transformations of λ together with the following semigroup operation:

$$x(\alpha\beta) = (x\alpha)\beta \text{ if } x \in \operatorname{dom}(\alpha\beta) = \{y \in \operatorname{dom}\alpha \colon y\alpha \in \operatorname{dom}\beta\}, \quad \text{for} \quad \alpha, \beta \in \mathscr{I}_\lambda.$$

The semigroup $\mathscr{I}_\lambda$ is called the *symmetric inverse semigroup* over the cardinal λ.

Definitions

Definition (Wagner–Clifford, 1952-1954)

A semigroup S is called *inverse* if for every $x \in S$ there exists a unique $y \in S$ such that $xyx = x$ and $yxy = y$. Such element y is said to be inverse of x and denote by x^{-1}. If S is an inverse semigroup, then the map $S \to S \colon x \mapsto x^{-1}$ is called *inversion*.

Definition

Put $\mathscr{I}_\lambda^n = \{\alpha \in \mathscr{I}_\lambda \colon \operatorname{rank} \alpha \leqslant n\}$, for $n = 1, 2, 3, \ldots$. Obviously, $\mathscr{I}_\lambda^n$ is an inverse subsemigroup of $\mathscr{I}_\lambda$, and moreover $\mathscr{I}_\lambda^n$ is an ideal of $\mathscr{I}_\lambda$, for each $n = 1, 2, 3, \ldots$. The semigroup $\mathscr{I}_\lambda^n$ is called the *symmetric inverse semigroup of finite transformations of the rank* $\leqslant n$.

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Definitions

Definitions

A *semitopological* (*topological*) *semigroup* is a Hausdorff topological space with separately continuous (continuous) semigroup operations. Inverse topological semigroup with continuous inversion is called a *topological inverse semigroup*.

A topology τ on a semigroup S is defined to be

- *shift-continuous* if for every $a \in S$ the left and right shifts $l_a \colon S \to S$, $l_a \colon x \mapsto ax$, and $r_a \colon S \to S$, $r_a \colon x \mapsto xa$, are continuous.
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Generalizations of compactness

compact | ω-bounded | ω-pracompact

sequentially compact | totally countably compact

countably compact | total. countably pracompact

sequentially pracompact | countably pracompact | H-closed

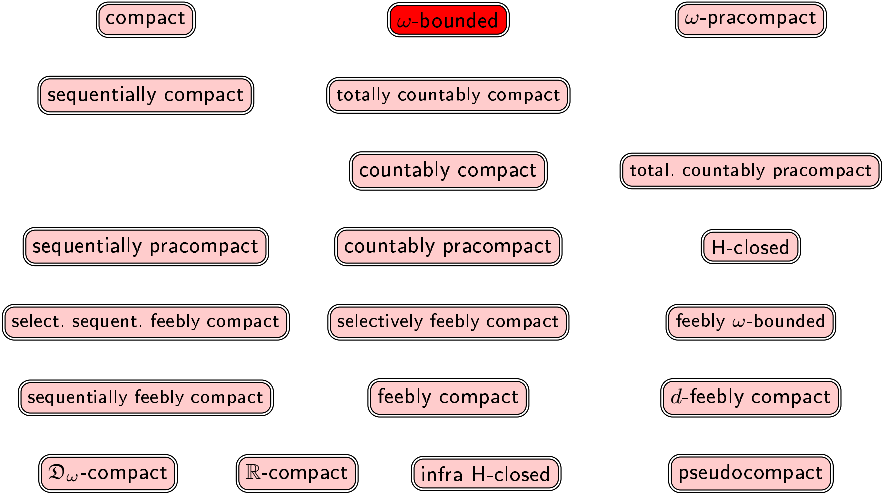
select. sequent. feebly compact | selectively feebly compact | feebly ω-bounded

sequentially feebly compact | feebly compact | d-feebly compact

$\mathfrak{D}_\omega$-compact | $\mathbb{R}$-compact | infra H-closed | pseudocompact

A topological space X is said to be *compact* if each open cover of X has a finite subcover.

Generalizations of compactness



A topological space X is said to be *ω-bounded* if each countable subset of X has the compact closure.

compact

ω-bounded

ω-pracompact

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totally countably compact

countably compact

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pseudocompact

A topological space X is said to be *ω-pracompact* if X contains a dense subset D such that each countable subset of D has the compact closure in X [G.-Ravsky, 2018].

Generalizations of compactness

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A topological space X is said to be *sequentially compact* if each sequence $\{x_i\}_{i\in\mathbb{N}}$ of X has a convergent subsequence in X.

Generalizations of compactness

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A topological space X is said to be *totally countably compact* if each sequence of X contains a subsequence with the compact closure.

Generalizations of compactness

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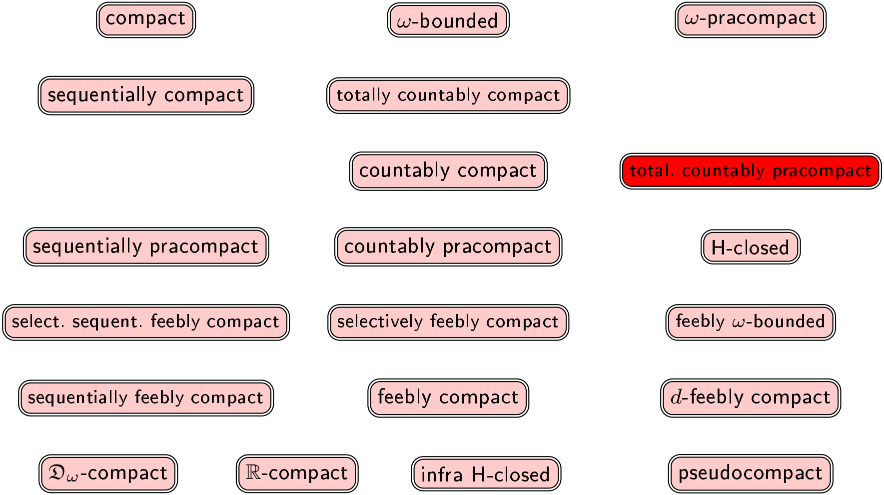
A topological space X is said to be *countably compact* if each open countable cover of X has a finite subcover.



A topological space X is said to be *totally countably pracompact* if there exists a dense subset D of X such that each sequence of points of D has a subsequence with the compact closure in X [G.-Ravsky, 2018].

Generalizations of compactness

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A topological space X is said to be *countably pracompact* if there exists a dense subset A in X such that X is countably compact at A [Arkhanegelskii, 1988].

Generalizations of compactness

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A topological space X is said to be *H-closed* if X is a closed subspace of every Hausdorff topological space in which it contained.

Generalizations of compactness

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A topological space X is said to be *selectively sequentially feebly compact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, one can choose a point $x_n \in U_n$ for every $n \in \mathbb{N}$ in such a way that the sequence $\{x_n : n \in \mathbb{N}\}$ has a convergent subsequence [Dorantes, Aldama, Shakhmatov, 2017].

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$\mathfrak{D}_\omega$-compact

$\mathbb{R}$-compact

infra H-closed

pseudocompact

A topological space X is said to be *selectively feebly compact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \varnothing\}$ is finite for every open neighborhood W of x [Dow, Porter, Stephenson, Woods, 2004].

Generalizations of compactness

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- feebly compact
- d-feebly compact
- $\mathfrak{D}_\omega$-compact
- $\mathbb{R}$-compact
- infra H-closed
- pseudocompact

A topological space X is said to be *feebly ω-bounded* if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of non-empty open subsets of X there is a compact subset K of X such that $K \cap U_n \neq \varnothing$ for each n [G.-Ravsky, 2018].

Generalizations of compactness

compact | ω-bounded | ω-pracompact

sequentially compact | totally countably compact

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sequentially pracompact | countably pracompact | H-closed

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sequentially feebly compact | feebly compact | d-feebly compact

$\mathfrak{D}_\omega$-compact | $\mathbb{R}$-compact | infra H-closed | pseudocompact

A topological space X is said to be *sequentially feebly compact* if for every family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X, there exists an infinite set $J \subseteq \mathbb{N}$ and a point $x \in X$ such that the set $\{n \in J : W \cap U_n = \varnothing\}$ is finite for every open neighborhood W of x [Dow, Porter, Stephenson, Woods, 2004].

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A topological space X is said to be *feebly compact* if each locally finite open cover of X is finite [Bagley, Connell, McKnight, Jr., 1958].

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$\mathfrak{D}_\omega$-compact | $\mathbb{R}$-compact | infra H-closed | pseudocompact

A topological space X is said to be *d-feebly compact* (or *DFCC*) if every discrete family of open subsets in X is finite.

Generalizations of compactness

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$\mathfrak{D}_\omega$-compact | $\mathbb{R}$-compact | infra H-closed | **pseudocompact**

A topological space X is said to be *pseudocompact* if X is Tychonoff and each continuous real-valued function on X is bounded.

Generalizations of compactness

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A topological space X is said to be *infra H-closed* provided that any continuous image of X into any first countable Hausdorff space is closed [Hajek, Todd, 1975].

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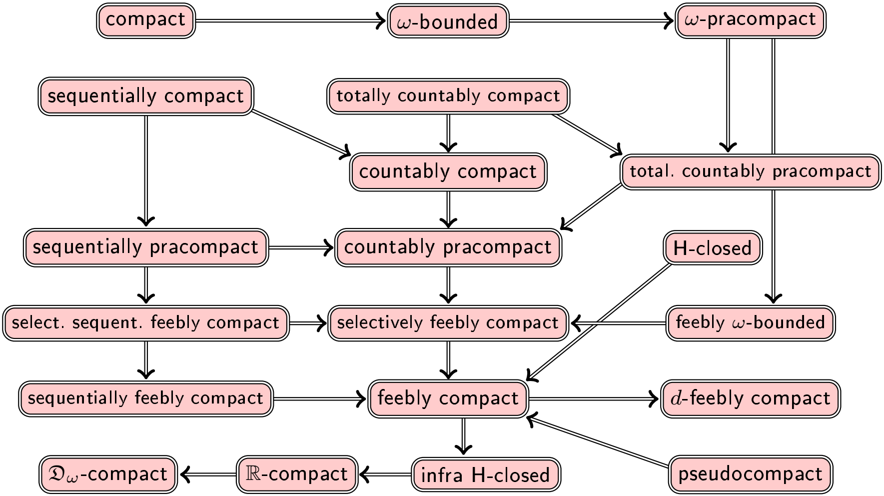
A topological space X is said to be *Y-compact* for some topological space Y, if $f(X)$ is compact, for any continuous map $f \colon X \to Y$.

Generalizations of compactness



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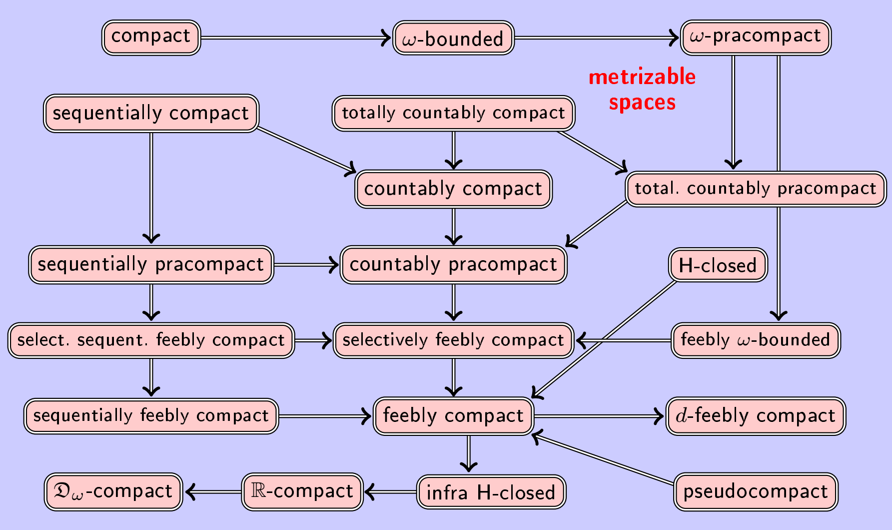

Old Results

Theorem (G-Pavlyk-Reiter, 2009)

For infinite cardinal λ the semigroup $\mathscr{I}_\lambda^1$ does not embed into a Hausdorff countably compact topological semigroup.

Theorem (G-Pavlyk, 2005)

For any infinite cardinal λ there exists a unique shift-continuous Hausdorff compact topology on $\mathscr{I}_\lambda^1$.

This topology is the Alexadroff one-point compactification of the discrete space of cardinality λ and it will denote by τ_{Ac}.

Theorem (G-Pavlyk, 2005)

Let λ by any infinite cardinal. Then every shift-continuous T_1-topology on $\mathscr{I}_\lambda^1$ is collectionwise normal and the following statement are equivalent:

(i) $(\mathscr{I}_\lambda^1, \tau)$ is a compact Hausdorff semitopological semigroup;

(ii) $\tau = \tau_{\mathsf{Ac}}$;

(iii) $(\mathscr{I}_\lambda^1, \tau)$ is a feebly compact Hausdorff semitopological semigroup.

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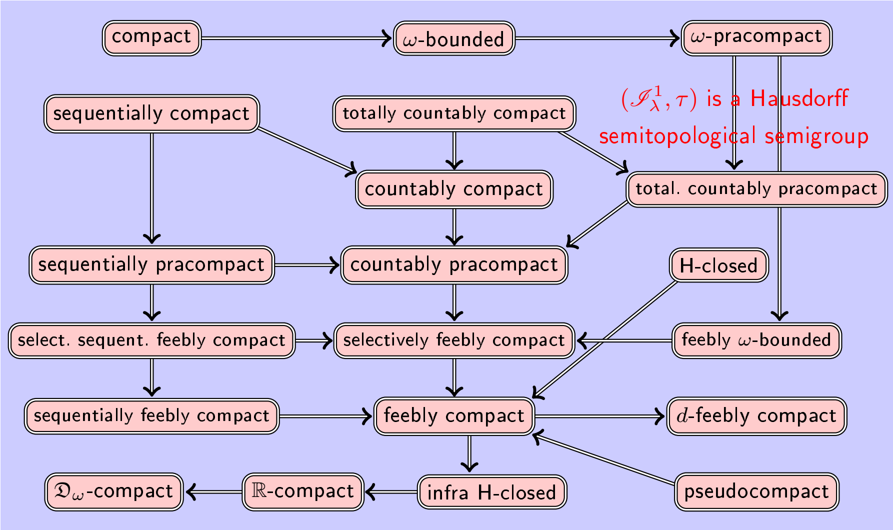
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pseudocompact

Old Results

Theorem (G-Lawson-Repovš, 2009)

Let λ be any infinite cardinal and n be any positive integer. If a Hausdorff semitopological semigroup S with continuous inversion contains $\mathscr{I}_\lambda^n$, then $\mathscr{I}_\lambda^n$ is a closed subsemigroup of S.

Theorem (G-Reiter, 2010)

Let λ be any infinite cardinal, n be any positive integer and $h\colon \mathscr{I}_\lambda^n \to S$ be homomorphism into a Hausdorff semitopological semigroup S with continuous inversion. Then $(\mathscr{I}_\lambda^n)h$ is a closed subsemigroup of S.

Definition

Let $\mathscr{S}$ be a class of Hausdorff semitopological semigroups. A semigroup $S \in \mathscr{S}$ is called *H-closed* in $\mathscr{S}$, if S is a closed subsemigroup of any topological semigroup $T \in \mathscr{S}$ which contains S both as a subsemigroup and as a topological space.

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Let λ by any infinite cardinal. Then a semitopological semigroup $(\mathscr{I}_\lambda^1, \tau)$ is H-closed in the class of Hausdorff semitopological semigroups if and only if $(\mathscr{I}_\lambda^1, \tau)$ is compact.

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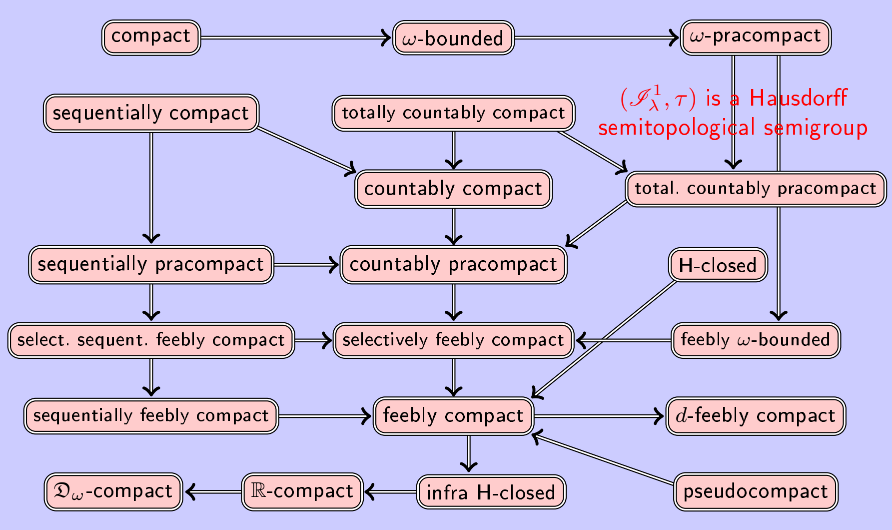

Any above condition is equivalent to the following: $(\mathscr{I}_\lambda^1, \tau)$ is H-closed in the class of Hausdorff semitopological semigroups.

Question

Do the above statements hold for a semitopological semigroup $(\mathscr{I}_\lambda^n, \tau)$ for $n > 1$?

Definition (Wagner, 1952)

On an inverse semigroup S we define the *natural partial order* $\preccurlyeq$ on S in the following way:

$$x \preccurlyeq y \quad \text{if and only if there exists an idempotent } e \in S \text{ such that } x = ey.$$

Example (G-Reiter, 2010)

Fix an arbitrary positive integer n. The following family

$$\mathscr{B}_{\mathsf{c}} = \{U_\alpha(\alpha_1, \ldots, \alpha_k) = {\uparrow_\preccurlyeq}\alpha \setminus ({\uparrow_\preccurlyeq}\alpha_1 \cup \cdots \cup {\uparrow}\alpha_k)_\preccurlyeq : \alpha_i \in {\uparrow_\preccurlyeq}\alpha \setminus \{\alpha\}, \alpha, \alpha_i \in \mathscr{I}_\lambda^n, i = 1, \ldots, k\}$$

determines a base of the topology τ_{c} on $\mathscr{I}_\lambda^n$. Then $(\mathscr{I}_\lambda^n, \tau_{\mathsf{c}})$ is Hausdorff compact semitopological semigroup with continuous inversion.

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Theorem

Let n be an arbitrary positive integer, λ be any infinite cardinal and τ be a T_1-shift continuous topology on the semigroup $\mathscr{I}_\lambda^n$. Then the following conditions are equivalent:

(i) τ is compact;

(ii) $\tau = \tau_{\mathsf{c}}$;

(iii) τ is countably compact;

(iv) τ is sequentially compact;

(v) τ is ω-pracompact;

(vi) τ is feebly ω-bounded.

Theorem

Let n be an arbitrary positive integer and λ be an arbitrary infinite cardinal. Then for every shift-continuous T_1-topology τ on the semigroup $\mathscr{I}_\lambda^n$ the following conditions are equivalent:

- (i) τ is sequentially pracompact;
- (ii) τ is total. countably pracompact;
- (iii) τ is d-feebly compact;
- (iv) $(\mathscr{I}_\lambda^n,\tau)$ is $\mathfrak{D}_\omega$-compact.

Example

For any infinite cardinal λ and any positive integer $n\geqslant 2$ there exists a Hausdorff feebly compact topology τ on the semigroup $\mathscr{I}_\lambda^n$ such that $(\mathscr{I}_\lambda^n,\tau)$ is a non-compact non-semiregular semitopological semigroup.

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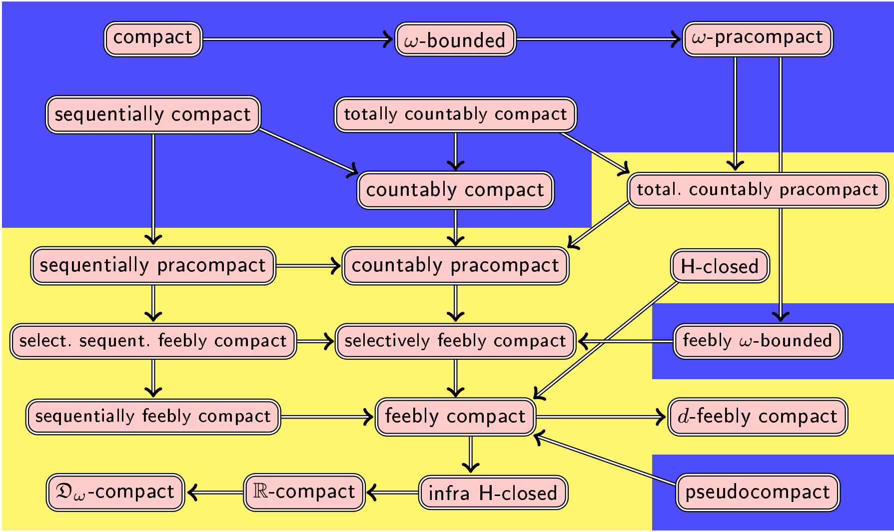
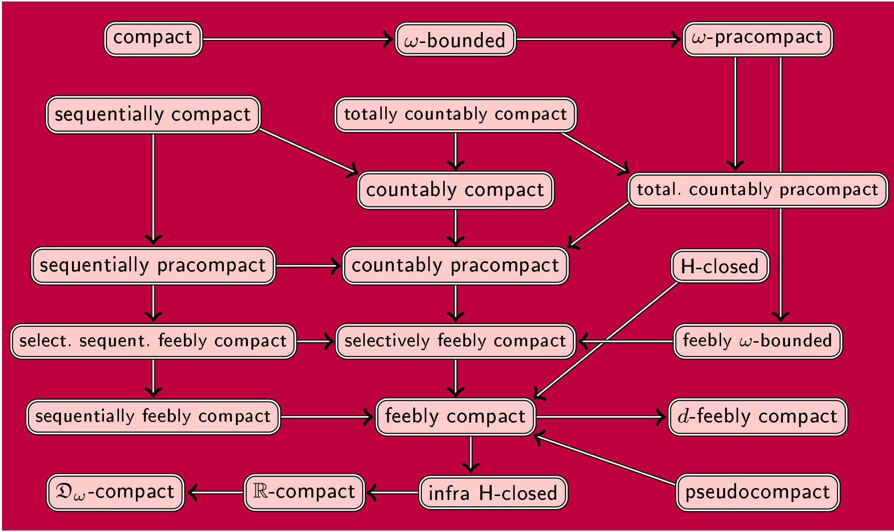

compact
ω-bounded
ω-pracompact
sequentially compact
totally countably compact
countably compact
total. countably pracompact
sequentially pracompact
countably pracompact
H-closed
select. sequent. feebly compact
selectively feebly compact
feebly ω-bounded
sequentially feebly compact
feebly compact
d-feebly compact
𝔇ω-compact
ℝ-compact
infra H-closed
pseudocompact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Thank You for attention!