Endomorphisms of totally disconnected groups

Helge Glöckner (Paderborn)

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First lecture: Automorphisms

§1 Scale $s(\alpha) \in \mathbb{N}$ and tidy subgroups for $\alpha \in \mathrm{Aut}(G)$, if G tot disc, lcp group

§2 Associated α-stable subgroups

§3 Automorphisms with closed contraction groups

§4 Contractive automorphisms

§5 Expansive automorphisms

§1 Tidy subgroups and scale

G totally disconnected, locally compact topological group

Theorem (van Dantzig 1936) *The compact open subgroups of G form a basis of identity neighbourhoods in G.*

None of them needs to be a normal subgroup.

Is it possible to achieve that a compact open subgroup U behaves nicely at least under conjugation with a single group element g, i.e. under the inner automorphism

$$\alpha_g\colon G \to G, \quad x \mapsto gxg^{-1}?$$

$\rightsquigarrow$ tidy subgroups for α_g (Willis 1994)

More generally, look for compact open subgroup which are well adapted to an automorphism $\alpha\colon G \to G$.

G tot disc, lcp group, $\alpha \in \mathrm{Aut}(G)$

Theorem (Willis 1994) *There exists a compact open subgroup U in G which is* **tidy for** α *in the sense that*

(TA) $U = U_+ U_-$, *where*

$$U_+ = \bigcap_{k \in \mathbb{N}_0} \alpha^k(U) \text{ and } U_- := \bigcap_{k \in \mathbb{N}_0} \alpha^{-k}(U)$$

are compact subgroups with

$$\alpha(U_+) = \bigcap_{k \geq 1} \alpha^k(U) \supseteq U_+$$

and $\alpha(U_-) = \bigcap_{k=0}^{\infty} \alpha^{1-k}(U) \subseteq U_-$;

(TB) $U_{++} = \bigcup_{k \in \mathbb{N}_0} \alpha^k(U_+)$ *is closed in* G.

Then $U_+ = U \cap \alpha(U_+)$ is open in the compact group $\alpha(U_+)$, whence the index

$$s(\alpha) := [\alpha(U_+) : U_+]$$

is finite.

Willis 1994: $s(\alpha) \in \mathbb{N}$ is independent of the choice of tidy subgroup; call $s(\alpha)$ the **scale of** α

U tidy:

- $U = U_+ U_-$ with $\alpha(U_+) \supseteq U_+$, $\alpha(U_-) \subseteq U_-$;
- $\bigcup_{k \in \mathbb{N}_0} \alpha^k(U_+)$ is closed in G

scale $s(\alpha) := [\alpha(U_+) : U_+]$

Properties (Willis 1994)

(a) $s(\alpha^n) = s(\alpha)^n$ for $n \in \mathbb{N}_0$;

(b) $s(\alpha) = s(\alpha^{-1}) = 1$ if and only if G has an α-stable compact open subgroup U

(i.e., $\alpha(U) = U$)

(c) Let mod$(\alpha) > 0$ be the module of α, i.e.

$$\lambda(\alpha(A)) = \text{mod}(\alpha)\lambda(A)$$

for Borel sets $A \subseteq G$ and a Haar measure λ. Then

$$\text{mod}(\alpha) = \frac{s(\alpha)}{s(\alpha^{-1})}.$$

(d) If U is tidy for α and $x \in U$, then $x \in U_-$ if and only if $\alpha^k(x) \in U$ for all $k \in \mathbb{N}_0$, if and only if $\{\alpha^k(x) \colon k \in \mathbb{N}_0\}$ is relatively compact.

Example (a) Right shift α on $F^{\mathbb{Z}}$

F a finite group
$G := F^{\mathbb{Z}}$ with product topology
$\alpha\colon G \to G$, $(x_n)_{n\in\mathbb{Z}} \mapsto (x_{n-1})_{n\in\mathbb{Z}}$

Since G is compact and α-stable, $U := G$ is a tidy subgroup for α with $U_+ = U_- = U$ and $s(\alpha) = s(\alpha^{-1}) = 1$.

By contrast,

$$V := \{(x_n)_{n\in\mathbb{Z}} \in G\colon x_0 = e\}$$

is a compact open subgroup of G with

$V_+ = F^{\{\ldots,-2,-1\}} \times \{e\}$ and $V_- = \{e\} \times F^{\{1,2,\ldots\}}$,
whence

$$V = V_+ V_- .$$

However,

$$V_{++} = \bigcup_{k\in\mathbb{N}_0} \alpha^k(V_+) = F^{-\mathbb{N}} \times F^{(\mathbb{N}_0)}$$

(with $F^{(\mathbb{N}_0)} := \bigoplus_{n\in\mathbb{N}_0} F$) is a dense proper subgroup of G and therefore not closed. Thus V is not tidy for α

(b) $\mathbb{Q}_p$ the field of p-adic numbers $z = \sum_{k=k_0}^{\infty} a_k p^k$ with $a_k \in \{0, \ldots, p-1\}$. If $a_{k_0} \neq 0$,

$$|z|_p := p^{-k_0}.$$

Has set $\mathbb{Z}_p \sim \{0, 1, \ldots, p-1\}^{\mathbb{N}_0}$ of p-adic integers $\sum_{k=0}^{\infty} a_k p^k$ as compact open subring.

Consider $G = \mathrm{GL}_2(\mathbb{Q}_p)$, $\alpha := \alpha_g$ with $g = \mathrm{diag}(1, p)$. Then

$$g^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} g^{-n} = \begin{pmatrix} a & p^{-n}b \\ p^n c & d \end{pmatrix}$$

Compact open subgroup

$$U := \mathbf{1} + (p\mathbb{Z}_p)^{2\times 2};$$

U_- lower triangular matrices in U,
U_+ upper triangular matrices in U
can show $U = U_+ U_-$,

$$U_{++} = \bigcup_{n \in \mathbb{N}_0} \alpha^n(U_+) = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Q}_p \\ 0 & 1 + p\mathbb{Z}_p \end{pmatrix}$$

closed in $\mathrm{GL}_2(\mathbb{Q}_p)$. Hence U tidy for $\alpha = \alpha_g$.

$$\alpha_g(U_+) = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 1 + p\mathbb{Z}_p \end{pmatrix}$$

implies $s(\alpha_g) = [\alpha_g(U_+) : U_+] = p$

Re-interpretation of scale and tidy subgroups

For $U \leq G$ c.o. consider **displacement index**

$$[\alpha(U) : \alpha(U) \cap U] \in \mathbb{N}$$

Theorem (Willis 2001) *U minimizes the displacement index iff U is tidy for α. The scale of α is the minimum displacement index,*

$$s(\alpha) = \min_U [\alpha(U) : \alpha(U) \cap U].$$

§2 Subgroups associated to $\alpha \in \mathrm{Aut}(G)$

Given a tot disc, lcp group G and $\alpha \in \mathrm{Aut}(G)$, have:

contraction group

$$\mathrm{con}(\alpha) := \{x \in G\colon \lim_{n\to\infty} \alpha^n(x) = e\}$$

and the anti-contraction group $\mathrm{con}(\alpha^{-1})$;

parabolic subgroup

$$\mathrm{par}(\alpha) := \{x \in G\colon \{\alpha^n(x)\colon n \in \mathbb{N}_0\} \text{ rel. cp.}\}$$

and anti-parabolic subgroup $\mathrm{par}(\alpha^{-1})$; and the

Levi subgroup $\mathrm{lev}(\alpha) := \mathrm{par}(\alpha) \cap \mathrm{par}(\alpha^{-1})$ of all $x \in G$ whose two-sided α-orbit $\{\alpha^n(x)\colon n \in \mathbb{Z}\}$ is relatively compact.

We shall see that $\mathrm{con}(\alpha)$ need not be closed. But $\mathrm{par}(\alpha)$, $\mathrm{par}(\alpha^{-1})$, $\mathrm{lev}(\alpha)$ always closed.

[If $U \subseteq G$ is tidy for α, then $U \cap \mathrm{par}(\alpha) = U_-$ is closed in G. Being locally closed, $\mathrm{par}(\alpha)$ is closed]

Example that con(α)$=\{x \in G\colon \lim_{n\to\infty} \alpha^n(x) = e\}$ need not be closed:

Consider the right shift α on $G := F^{\mathbb{Z}}$ as above. Then con(α) $= F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ is a proper dense subgroup, hence not closed.

[Note $\alpha^n((x_k)_{k\in\mathbb{Z}}) \in F^{-\mathbb{N}} \times \{e\} \times F^{\mathbb{N}}$ for $n \geq n_0$ implies $x_k = e$ for all $k \leq -n_0$]

Remark (a) If G has a compact open subgroup satisfying the ascending chain condition on closed subgroups, then con(α) is closed in G for each $\alpha \in \text{Aut}(G)$. For instance, this applies to each p-adic Lie group

(b) For $F = \mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, the shift example can be interpreted as a 2-dimensional Lie group over the local field $\mathbb{F}_p((t))$ of formal Laurent series; the shift α is locally linear and hence analytic. Non-closed contraction groups can occur in positive characteristic!

§3 Automorphisms with closed contraction groups

Baumgartner/Willis (2004):

- $s(\alpha) = \text{mod}(\alpha|_{\overline{\text{con}(\alpha^{-1})}})$;

- $\text{con}(\alpha)$ is closed if and only if $\text{con}(\alpha^{-1})$ is closed, if and only if G has small tidy subgroups for α

In this case,

$$\text{par}(\alpha) = \text{con}(\alpha) \rtimes \text{lev}(\alpha).$$

(Metrizability removed by Jaworski 2009)

Corollary (G'05) *If* $\text{con}(\alpha)$ *is closed, then* $\text{con}(\alpha)\,\text{lev}(\alpha)\,\text{con}(\alpha^{-1})$ *is open in* G *and the product map*

$$\text{con}(\alpha)\times\text{lev}(\alpha)\times\text{con}(\alpha^{-1}) \to \text{con}(\alpha)\,\text{lev}(\alpha)\,\text{con}(\alpha^{-1}),$$

$(x, y, z) \mapsto xyz$, *is a homeomorphism.*

(p-adic case: Wang 1984).

Cor. (G'05) *If* con(α) *is closed, then* con(α) lev(α) con(α^{-1}) *is open in* G *and the product map*

$\pi\colon$ con(α) $\times$ lev(α) $\times$ con(α^{-1}) $\to$ con(α) lev(α) con(α^{-1}),

$(x, y, z) \mapsto xyz$, *is a homeomorphism.*

Remark If α is an analytic automorphism of a Lie group G over a totally disconnected local field $\mathbb{K}$ and con(α) is closed, then con(α), con(α^{-1}) and lev(α) are Lie subgroups of G and the product map π is an analytic diffeomorphism (G.)

Have stable/unstable foliations

Proof uses invariant manifolds as a tool (stable, unstable, resp., centre manifolds around a hyperbolic fixed point), constructed in G'13 for time-discrete, $\mathbb{K}$-analytic dynamical systems

Closedness of contraction groups is also important in connection with topological entropy.

Theorem (Giordano Bruno/Virili 2017) *If G is a totally disconnected, locally compact group, $\alpha \in \mathrm{Aut}(G)$ and $\mathrm{con}(\alpha)$ is closed, then*

$$h_{\mathrm{top}}(\alpha) = \ln s(\alpha)$$

(likewise for endomorphisms)

§4 Contractive automorphisms

If con(α) is closed, then con(α) is a locally compact group and $\alpha|_{\mathrm{con}(\alpha)}$ is a contractive automorphism:

Definition An automorphism α of a locally compact group G is **contractive** if $\alpha^n(x) \to e$ as $n \to \infty$ for each $x \in G$ (i.e., $G = \mathrm{con}(\alpha)$).

What is the structure of such contraction groups (G, α)?

Theorem (Siebert 1986) *If α is a contractive automorphism of a locally compact group G, then $G = G_e \times G_{\mathrm{td}}$, where G_e is the connected component and G_{td} an α-stable, totally disconnected closed normal subgroup of G.*

Siebert also showed that G_e is a simply connected nilpotent real Lie group whose Lie algebra admits a positive graduation.

Structure of G_{td}?

Theorem (G./Willis 2010) *If* (G, α) *is a totally disconnected locally compact contraction group, then the set* $\text{tor}(G)$ *of torsion elements is a closed subgroup of* G *and*

$$G = \text{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}$$

with certain α*-stable closed normal subgroups* $G_p \subseteq G$ *which are* p*-adic Lie groups.*

Remark p-adic contraction groups are unipotent algebraic groups over $\mathbb{Q}_p$ (Wang 1984); they correspond to nilpotent p-adic Lie algebras admiting an $\mathbb{N}$-graduation.

Structure of $\text{tor}(G)$?

G./Willis 2010: $\text{tor}(G)$ is a torsion group of finite exponent and has a composition series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of α-stable closed subgroups all of whose factors G_j/G_{j-1} are isomorphic to the right shift on $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ for some finite simple group F (depending on j)

Only countably many possible composition factors. Does this imply there are only countably many torsion contraction groups?

Theorem (G./Willis 2018) *Suitable continuous equivariant 2-cocycles*

$$\omega\colon \mathbb{F}_p((t)) \times \mathbb{F}_p((t)) \to \mathbb{F}_p((t))$$

yield a family of uncountably many contraction groups of the form

$$\mathbb{F}_p((t)) \times_\omega \mathbb{F}_p((t))$$

which are pairwise non-isomorphic as contraction groups.

However, up to isomorphism there are only countably many **abelian** torsion contraction groups, as any such is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift for some finite abelian group F (loc. cit.)

All quotient morphisms $q\colon G \to H$ of locally compact contraction groups admit an equivariant global section, whence central extensions (or extensions with abelian kernel) can be described by equivariant continuous 2-cocycles and corresponding cohomology groups (loc. cit.)

§5 Expansive automorphisms

Definition An automorphism α of a locally compact group G is called **expansive** if there exists an identity neighbourhood $V \subseteq G$ such that

$$\bigcap_{n \in \mathbb{Z}} \alpha^n(V) = \{e\}.$$

For example, every contractive automorphism is expansive.

Studied e.g. by Siebert 1989.

Theorem G./Raja (2017) *If G is a totally disconnected, locally compact group and α an expansive automorphism, then* $\mathrm{con}(\alpha)\,\mathrm{con}(\alpha^{-1})$ *is an open identity neighbourhood. For every α-stable closed normal subgroup $N \subseteq G$, the induced automorphism of G/N is expansive.*

Summary

- For $\alpha \in \text{Aut}(G)$, Willis theory provides tidy subgroups $U = U_+U_-$ and the scale $s(\alpha) = [\alpha(U_+) : U_+]$

- Have α-stable subgroups con(α), con(α^{-1}), par(α), par(α^{-1}), lev(α)

- Most information on dynamics of (G, α) if con(α) is closed

- Results known on structure of closed contraction groups con(α)

Endomorphisms of totally disconnected groups

Helge Glöckner (Paderborn)

Udine, July 5, 2018

Second lecture: Endomorphisms

§1 Scale $s(\alpha) \in \mathbb{N}$ and tidy subgroups for $\alpha \in \mathrm{End}(G)$, if G tot disc, lcp group

§2 Subgroups associated with α

§3 Example: Linear endomorphisms

§4 Example: p-adic Lie groups and Lie groups over local fields

For a totally disconnected locally compact group G and $\alpha \in \mathrm{Aut}(G)$, recall:

If $U \subseteq G$ is a compact open subgroup, obtain compact subgroups $U_+, U_- \subseteq U$ with

$$\alpha(U_-) \subseteq U_-, \quad \alpha(U_+) \supseteq U_+$$

via

$$U_- := \bigcap_{n \in \mathbb{N}_0} \alpha^{-n}(U) = \{x \in U \colon (\forall n \in \mathbb{N}_0)\, \alpha^n(x) \in U\}$$

$$U_+ := \bigcap_{n \in \mathbb{N}_0} \alpha^{n}(U) = \{x \in U \colon (\forall n \in \mathbb{N}_0)\, \alpha^{-n}(x) \in U\}$$

Thus $U_+ \subseteq \alpha(U_+) \subseteq \alpha^2(U_+) \subseteq \cdots$

Willis 1994: There always exists a compact open subgroup U which is **tidy for** α, i.e.

(TA) $U = U_+ U_-$

(TB) $U_{++} := \bigcup_{n \in \mathbb{N}_0} \alpha^n(U_+)$ is closed in G

(equivalently, $U_{--} := \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(U_-)$ is closed)

Analogues for endomorphisms?

For $\alpha \in \text{End}(G)$ and a compact open subgroup U of G,

$$U_- := \bigcap_{n\in\mathbb{N}_0} \alpha^{-n}(U) = \{x \in U \colon (\forall n \in \mathbb{N}_0)\, \alpha^n(x) \in U\}$$

is a compact subgroup of U with $\alpha(U_-) \subseteq U_-$.

As to U_+, need replacement for $\alpha^{-n}(x)$ if α is not an automorphism:

Call a sequence $(x_{-n})_{n\in\mathbb{N}_0}$ in G an **α-regressive trajectory** if $\alpha(x_{-n-1}) = x_{-n}$ for all $n \in \mathbb{N}_0$.

Define U_+ as the set of all $x \in U$ for which there exists an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ in U with $x_0 = x$.

Note that $x_{-1} \in U_+$, whence $x = x_0 = \alpha(x_{-1}) \in \alpha(U_+)$ and thus

$$\alpha(U_+) \supseteq U_+; \quad \text{in fact, } \alpha(U_+) \cap U = U_+.$$

Call a compact open subgroup $U \subseteq G$ **tidy** for $\alpha \in \text{End}(\alpha)$ if

(TA) $U = U_+U_-$

(TB) $U_{--} := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(U_-)$ is closed in G

Call a sequence $(x_{-n})_{n\in\mathbb{N}_0}$ in G an **α-regressive trajectory** if $\alpha(x_{-n-1}) = x_{-n}$ for all $n \in \mathbb{N}_0$.

Define U_+ as the set of all $x \in U$ for which there exists an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ in U with $x_0 = x$.

Lemma. *U_+ is compact*

Proof. Show that U_+ is closed in U. To this end, let $(x_j)_{j\in J}$ be a net in U_+ which converges to some $x \in U$. Let $t_j := (x_{j,-n})_{n\in\mathbb{N}_0} \in U^{\mathbb{N}_0}$ be an α-regressive trajectory with $x_{j,0} = x_j$. Since $U^{\mathbb{N}_0}$ is compact, after passage to a subnet we may assume that $t_j \to (x_{-n})_{n\in\mathbb{N}_0}$ for some $(x_{-n})_{n\in\mathbb{N}_0} \in U^{\mathbb{N}_0}$. Passing to the limit in

$$\alpha(x_{j,-n-1}) = x_{j,-n},$$

see that $(x_{-n})_{n\in\mathbb{N}_0}$ is an α-regressive trajectory. As $x_0 = \lim x_{j,0} = \lim x_j = x$, have $x \in U_+$.

Defn U is **tidy for** α if

(TA) $U = U_+ U_-$, *where*

$U_- = \{x \in U : \alpha^n(x) \in U$ for all $n \in \mathbb{N}_0\}$

$U_+ = \{x_0 \in U : \exists (x_{-n})$ in U

$$(\forall n \in \mathbb{N}_0)\ \alpha(x_{-n-1}) = x_{-n}\}$$

(TB) $U_{--} = \bigcup_{n \in \mathbb{N}_0} \alpha^{-n}(U_-)$ *is closed in* G.

Scale $s(\alpha) := [\alpha(U_+) : U_+] \in \mathbb{N}$

Note that

- $U_- = \bigcap_{n \in \mathbb{N}_0} \alpha^{-n}(U)$ compact
- U_{--} ascending union as $\alpha^{-n}(U_-)$ is the set of all $x \in G$ such that $\alpha^k(x) \in U$ for all $k \geq n$
- $\alpha(U_+) \supseteq U_+$ and $\alpha(U_-) \subseteq U_-$ by definition
- $U_+ = \alpha(U_+) \cap U$ is open in $\alpha(U_+)$

Theorem (Willis 2015) *For every endomorphism* α *of a totally disconnected locally compact group* G*, there exists a compact open subgroup of* G *which is tidy for* α*. The scale* $s(\alpha)$ *is independent of the choice of tidy subgroup. Moreover,*

$$s(\alpha) = \min_U [\alpha(U) : \alpha(U) \cap U]$$

is the minimum displacement index, and a compact open subgroup $U \subseteq G$ *is tidy for* α *if and only if* $s(\alpha) = [\alpha(U) : \alpha(U) \cap U]$*.*

§2 Subgroups associated to an endomorphism (Willis 2015)

For tot disc, lcp grp G and $\alpha \in \mathrm{End}(G)$, have:

contraction group

$$\mathrm{con}(\alpha) := \{x \in G \colon \lim_{n\to\infty} \alpha^n(x) = e\}$$

anti-contraction group $\mathrm{con}^-(\alpha)$ of all $x \in G$ such that $x = x_0$ for some α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ with $\lim_{n\to\infty} x_{-n} = e$

parabolic subgroup

$$\mathrm{par}(\alpha) := \{x \in G \colon \{\alpha^n(x) \colon n \in \mathbb{N}_0\} \text{ rel. cp.}\}$$

anti-parabolic subgroup $\mathrm{par}^-(\alpha)$ of all $x \in G$ such that $x = x_0$ for some α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ with $\{x_{-n} \colon n \in \mathbb{N}_0\}$ rel. compact

Levi subgroup $\mathrm{lev}(\alpha) := \mathrm{par}(\alpha) \cap \mathrm{par}^-(\alpha)$

If α is an automorphism, simply

$\mathrm{con}^-(\alpha) = \mathrm{con}(\alpha^{-1})$ and $\mathrm{par}^-(\alpha) = \mathrm{par}(\alpha^{-1})$.

con(α) $:= \{x \in G \colon \text{lim}_{n\to\infty} \alpha^n(x) = e\}$

par(α) $:= \{x \in G \colon \{\alpha^n(x) \colon n \in \mathbb{N}_0\}$ rel. cp.$\}$

lev(α) := par(α) $\cap$ par$^-(\alpha)$

con$^-(\alpha)$ set of all $x \in G$ such that $x = x_0$ for some α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ with $\lim\limits_{n\to\infty} x_{-n} = e$

par$^-(\alpha)$ set of $x \in G$ such that $x = x_0$ for an α-regressive trajectory $(x_{-n})_{n\in\mathbb{N}_0}$ with $\{x_{-n} \colon n \in \mathbb{N}_0\}$ rel. comp.

As in the case of automorphisms, par(α), par$^-(\alpha)$ and lev(α) are closed. Moreover,

$$\alpha(\text{con}(\alpha)) \subseteq \text{con}(\alpha), \quad \alpha(\text{par}(\alpha)) \subseteq \text{par}(\alpha),$$

$$\alpha(\text{con}^-(\alpha)) = \text{con}^-(\alpha), \quad \alpha(\text{par}^-(\alpha)) = \text{par}^-(\alpha)$$

and $\quad \alpha(\text{lev}(\alpha)) = \text{lev}(\alpha)$.

Need not have $\alpha(\text{con}(\alpha)) = \text{con}(\alpha)$:

For example, $\alpha(\text{con}(\alpha)) = \{e\} \subset \text{con}(\alpha) = G$ if $\alpha \colon G \to G$, $x \mapsto e$

Recall facts which are valid for automorphisms $\alpha \in \mathrm{Aut}(G)$:

- $\mathrm{con}(\alpha)$ is closed if and only if $\mathrm{con}^-(\alpha)$ is closed
- $\mathrm{con}(\alpha)$ is closed if and only if G has small tidy subgroups for α; in this case, the product map

$$\mathrm{con}(\alpha)\times\mathrm{lev}(\alpha)\times\mathrm{con}^-(\alpha) \to \mathrm{con}(\alpha)\,\mathrm{lev}(\alpha)\,\mathrm{con}^-(\alpha),$$

$(x, y, z) \mapsto xyz$, is a homeomorphism.

Open problems during "Winter of Disconnectedness" (Melbourne + Newcastle) in 2016:

Do these facts carry over to endomorphisms α?

Left-shift α on $\mathbb{F}^{\mathbb{N}}$ shows closedness of $\mathrm{con}^-(\alpha) = F^{\mathbb{N}}$ does not imply closedness of $\mathrm{con}(\alpha) = F^{(\mathbb{N})}$.

Theorem (Bywaters/G./Tornier 2018) *For* $\alpha \in$ *End*(G)*, the contraction group is closed if and only if* G *has small tidy subgroups for* α*. In this case, also* $\mathrm{con}^{-}(\alpha)$ *is closed and the product map*

$$\mathrm{con}(\alpha)\times\mathrm{lev}(\alpha)\times\mathrm{con}^{-}(\alpha) \to \mathrm{con}(\alpha)\,\mathrm{lev}(\alpha)\,\mathrm{con}^{-}(\alpha)$$

is a homeomorphism.

§3 Scale and tidy subgroups for $\alpha \in \mathrm{End}_{\mathbb{K}}(\mathbb{K}^n)$

$\mathbb{K}$ a local field (loc comp, tot disc, non-discrete)

$|.|_{\mathbb{K}}$ natural absolute value, for $x \in \mathbb{K}^\times$

$$|x|_{\mathbb{K}} = \ \mathrm{mod}_{\mathbb{K}}(m_x),\ m_x\colon \mathbb{K} \to \mathbb{K},\ y \mapsto xy$$

Same symbol for extension to abs value on $\overline{\mathbb{K}}$

ultrametric inequality:

$$|x+y|_{\mathbb{K}} \le \max\{|x|_{\mathbb{K}}, |y|_{\mathbb{K}}\}$$

E fin dim $\mathbb{K}$-vector space

If characteristic polynomial of $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$ splits into linear factors over $\mathbb{K}$, then E is the direct sum of the generalized eigenspaces.
For $\rho \in [0,\infty[$, let

$$E_\rho$$

be the sum of those for eigenvalues λ with $|\lambda|_{\mathbb{K}} = \rho$; thus

$$E = \bigoplus_{\rho \ge 0} E_\rho$$

General endomorphism:

Same in $E \otimes_{\mathbb{K}} \mathbb{L}$ with $\mathbb{L}/\mathbb{K}$, then intersect with E to get E_ρ

Fact *On the direct sum*

$$E = \bigoplus_{\rho \geq 0} E_\rho$$

of the α*-invariant characteristic subspaces, there exists an ultrametric norm* $\|.\|$ *on* E *which is a maximum norm w.r.t. the direct sum, such that* $\alpha|_{E_0}$ *is nilpotent,* $\|\alpha|_{E_0}\|_{op} < 1$ *and*

$$(\forall \rho > 0,\ v \in E_\rho)\ \ \|\alpha(v)\| = \rho\|v\|.$$

(**Reduce to Jordan blocks in** $E \otimes_{\mathbb{K}} \mathbb{L}$**!**)

The restriction of α to the "Fitting 1-component" $F := \bigoplus_{\rho > 0} E_\rho$ is an automorphism, and

$$E = E_0 \oplus F.$$

Thus

$$s(\alpha) = s(\alpha|_{E_0})s(\alpha|_F) = s(\alpha|_F)$$

Two cases:

1) α nilpotent (then $s(\alpha) = 1$)

2) α is an automorphism

Lemma *Let G be a totally disconnected, locally compact group and $\alpha \in \mathrm{End}(G)$.*

(a) *If α is nilpotent (say $\alpha^n = e$) and $U \subseteq G$ a compact open subgroup, then*

$$V := U_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(U) = \bigcap_{k=0}^{n-1} \alpha^{-k}(U)$$

is a compact open sgp of G with $\alpha(V) \subseteq V$.

(b) *If $V \subseteq G$ is a compact open subgroup such that $\alpha(V) \subseteq V$, then V is tidy, $V_- = V$ and $s(\alpha) = 1$.*

Proof. (a) clear. (b):

$V \subseteq \alpha^{-1}(V)$, hence $V \subseteq \alpha^{-k}(V)$ for all $k = 0, 1, \ldots$, hence

$$V_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(V) = V.$$

Hence $V = V_+ V_-$ (TA). As the subgroup

$$V_{--} = \bigcup_{k=0}^{\infty} \alpha^{-k}(V_-) = \bigcup_{k=0}^{\infty} \alpha^{-k}(V)$$

contains V, it is open and hence closed (TB).

As $\alpha(V) \subseteq V$, $\;s(\alpha) = [\alpha(V) : \underbrace{\alpha(V) \cap V}_{=\alpha(V)}] = 1$

Hence, for $\alpha \in \mathrm{End}_{\mathbb{K}}(E)$, $s(\alpha|_{E_0}) = 1$ and $B_r^{E_0}(0)$ is tidy for $\alpha|_{E_0}$ for each $r > 0$.

Now consider $\alpha \in \mathrm{GL}(E)$. Then

$$U := B_r^E(0) = \prod_{\rho > 0} B_r^{E_\rho}(0)$$

is tidy: For each $k \in \mathbb{Z}$, have

$$\alpha^k(B_r^E(0)) = \prod_{\rho > 0} B_{\rho^k r}^{E_\rho}(0).$$

Hence

$$U_+ := \bigcap_{k=0}^{\infty} \alpha^k(U) = \prod_{\rho \geq 1} B_r^{E_\rho}(0),$$

$$U_- = \bigcap_{k=0}^{\infty} \alpha^{-k}(U) = \prod_{0 < \rho \leq 1} B_r^{E_\rho}(0)$$

with $U = U_+ + U_-$ and

$$U_{--} := \bigcup_{k=0}^{\infty} \alpha^{-k}(U_-) = \prod_{0 < \rho < 1} E_\rho \times B_r^{E_1}(0)$$

closed in E.

Abbreviate $E_+ := \prod_{\rho \geq 1} E_\rho$. Then

$$s(\alpha) = [\alpha(U_+) : U_+] = \quad \mathrm{mod}_{E_+}(\alpha|_{E_+})$$

$$= \prod_{j:|\lambda_j|_{\mathbb{K}} \geq 1} |\lambda_j|_{\mathbb{K}}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of $\alpha_{\overline{\mathbb{K}}}$ in $\overline{\mathbb{K}}$ (with repetitions according to algebraic multiplicities).

Analytic functions, manifolds and Lie groups

(cf. Serre's book)

$\mathbb{K}^n$ with maximum norm;

for $\alpha \in \mathbb{N}_0^n$ a multi-index,

$y^\alpha := y_1^{\alpha_1} \cdots y_n^{\alpha_n}$ for $y = (y_1, \ldots, y_n) \in \mathbb{K}^n$.

$f \colon \mathbb{K}^n \supseteq U \to \mathbb{K}^m$ **analytic** if it is given locally by an absolut convergent power series around each point $x \in U$, i.e.,

$$f(x+y) = \sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \, y^\alpha \quad \text{for } y \in \mathbb{K}^n \text{ close to } 0$$

with $a_\alpha \in \mathbb{K}^m$.

Facts: compositions of analytic functions are analytic

$\rightsquigarrow$can define an n-dimensional analytic manifold M over a local field $\mathbb{K}$ as usual

(Hausdorff topological space M, equipped with a set $\mathcal{A}$ of homeomorphisms ϕ from open subsets of M onto open subsets of $\mathbb{K}^n$ such that the transition map $\psi \circ \phi^{-1}$ is analytic, for all $\phi, \psi \in \mathcal{A}$)

Analytic maps between analytic manifolds as usual (check analyticity in local charts).

A *Lie group* over a local field $\mathbb{K}$ is a group G with a (finite-dimensional) analytic manifold structure turning the group multiplication

$$m\colon G \times G \to G\,, \quad m(x,y) := xy$$

and the group inversion

$$j\colon G \to G\,, \quad j(x) := x^{-1}$$

into analytic mappings.

Examples

∗ additive groups of fin-dim $\mathbb{K}$-vector spaces

∗ general linear group $\mathrm{GL}_n(\mathbb{K}) = \det^{-1}(\mathbb{K}^\times)$ open in $\mathbb{K}^{n^2}$, group operations rational maps

∗ Every (group of $\mathbb{K}$-rational points of a) linear algebraic group defined over $\mathbb{K}$ is a $\mathbb{K}$-analytic Lie group, viz. every subgroup $G \leq \mathrm{GL}_n(\mathbb{K})$ which is the set of joint zeros of a set of polynomial functions $M_n(\mathbb{K}) \to \mathbb{K}$. E.g. $\mathrm{SL}_n(\mathbb{K})$

The Lie algebra functor. Tangent space

$$L(G) := T_e(G)$$

is a Lie algebra via the identification of $x \in L(G)$ with corresponding left invariant vector field on G (which form a Lie subalgebra of Lie algebra $\mathcal{V}^\omega(G)$ of analytic vector fields on G).

$\alpha\colon G \to H$ analytic group homomorphism between $\mathbb{K}$-analytic Lie groups, then tangent map

$$L(\alpha) := T_e(\alpha)\colon L(G) \to L(H)$$

is a Lie algebra homomorphism.

Every p-adic Lie group G has an **exponential function** $\exp_G\colon U \to G$ defined on a compact open subgroup $U \subseteq L(G)$. It is an analytic diffeomorphism onto an open identity neighbourhood such that $T_e \exp = \mathrm{id}_{L(G)}$ and $\exp(nx) = \exp(x)^n$ for all $x \in U$, $n \in \mathbb{Z}$.

If $\alpha \in \mathrm{End}(G)$, then

$$\alpha \circ \exp_G = \exp_G \circ L(\alpha)$$

on some 0-neighbourhood

The dynamical systems (G, α) and $(L(G), L(\alpha))$ are locally conjugate!

Theorem (G.) *For each endomorphism α of a p-adic Lie group G,*

$$s(\alpha) = s(L(\alpha))$$

holds. Moreover, $\exp_G(B_r(0))$ *is tidy for each ball $B_r(0) \subseteq L(G)$ with respect to a norm adapted to $L(\alpha)$ and $r > 0$ sufficiently small.*

Similarly for Lie groups over local fields:

Theorem (G.) *For an analytic endomorphism α of a Lie group G over a local field of positive characteristic, $s(\alpha) = s(L(\alpha))$ holds if and only if* $\mathrm{con}(\alpha)$ *is closed. In this case,* $\mathrm{con}(\alpha)$, $\mathrm{lev}(\alpha)$ *and* $\mathrm{con}^-(\alpha)$ *are Lie subgroups of G and the product map*

$$\mathrm{con}(\alpha)\times\mathrm{lev}(\alpha)\times\mathrm{con}^-(\alpha) \to \mathrm{con}(\alpha)\,\mathrm{lev}(\alpha)\,\mathrm{con}^-(\alpha)$$

is an analytic diffeomorphism. Moreover, $\phi^{-1}(B_r(0))$ is tidy in this case for each ball $B_r(0) \subseteq L(G)$ with respect to a norm adapted to $L(\alpha)$ and $r > 0$ sufficiently small, when ϕ is a chart for G around e with $\phi(e) = 0$ and $T_e\phi = \mathrm{id}_{L(G)}$.