Equivariant Choi-Effros lifting theorem and its applications

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joint work with Eusebio Gardella and Klaus Thomsen

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Outline

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- A linear map $\varphi \colon A \to B$ between C^*-algebras called completely positive (c.p) if the induced map $\varphi_n \colon M_n(A) \to M_n(B)$ defined by

$$\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$$

is positive for any $n \in \mathcal{N}$.

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  - A map $\varphi\colon A \to M_n(\mathbb{C})$ is c.p if and only if linear functional $\widetilde{\varphi}\colon M_n(A) \to \mathbb{C}$ defined by

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      is positive.
    - A map $\varphi\colon M_n(\mathbb{C}) \to A$ is c.p if and only if $[\varphi(e_{i,j})]$ is positive in $M_n(A)$.

- Let $\pi\colon A \to B(H)$ be a representation of C^*-algebra A on a Hilbert space H, and let $V \in B(H)$. Then linear map $\varphi\colon A \to B(H)$ defined by

$$\varphi(.) = V\pi(.)V^*$$

is a c.p.c map.

Definition. A map $\theta\colon A \to B$ is called nuclear if there exist c.p.c maps $\varphi_i\colon A \to M_{n_i}$ and $\psi_i\colon M_{n_i} \to B$ such that

$$\|\theta(a) - \psi_i \circ \varphi_i(a)\| \to 0$$

for all $a \in A$.

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Choi-Effros lifting theorem

Every nuclear c.p.c map from a separable C^*-algebra to a quotient map B/J is liftable.

Equivariant Choi-Effors lifting theorem, compact group actions

Theorem. Let G be a compact group, let A and B be C^*-algebras with A separable, let $\alpha\colon G \to Aut(A)$ and $\beta\colon G \to Aut(B)$ be actions, and let I be a β-invariant ideal in B. Denote by $q\colon A \to B/I$ the canonical quotient map, and by $\overline{\beta}\colon G \to Aut(B/I)$ the induced action.

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for any $g \in K$ and $a \in A$.

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Moreover we can choose σ to be unital when ϕ is unital.

Equivariant Choi-Effros lifting theorem, amenable group actions

Theorem. Let G be an amenable countable group, let A and B be C^*-algebras with A separable, let $\alpha\colon G \to Aut(A)$ and $\beta\colon G \to Aut(B)$ be actions, and let I be a β-invariant ideal in B. Denote by $q\colon A \to B/I$ the canonical quotient map, and by $\overline{\beta}\colon G \to Aut(B/I)$ the induced action.
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Equivariant Choi-Effros lifting theorem, general case

F-Gardella-Thomsen

Let G be a second countable locally compact group, let A and B be C^*-algebras with A separable, let $\alpha\colon G \to Aut(A)$ and $\beta\colon G \to Aut(B)$ be actions, and let I be a β-invariant ideal in B. Denote by $q\colon A \to B/I$ the canonical quotient map. Let $\varphi\colon A \to B/I$ be a nuclear completely positive contractive map.

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$$\sup_{a\in F}\sup_{g\in K} \|(\psi \circ \alpha_g)(a) - (\beta_g \circ \psi)(a)\| <$$

$$\eta + \sup_{a\in F}\sup_{g\in K} \|(\varphi \circ \alpha_g)(a) - (\overline{\beta}_g \circ \varphi)(a)\|.$$

Strongly self-absorbing C^*-algebras

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Tom and Winter study an abstract property of these algebras: *UHF*-algebras, $\mathcal{O}_2$, $\mathcal{O}_\infty$, and $\mathcal{Z}$ are all isomorphic to their tensor square in a **strong sense**.

DEFINITION. Let $\varphi_i \colon A \to B$ are $c.c.p$ maps, for $i = 1, 2$. We say that φ_1 and φ_2 are approximately unitarly (a.u) equivalent, $\varphi_1 \simeq_{a.u} \varphi_2$ if there exists a $(u_n)_{n\in\mathbb{N}} \subset \mathcal{U}(M(B))$ such that

$$\|u_n^*\varphi_2(a)u_n - \varphi_1(a)\| \to 0,$$

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DEFINITION. (Toms-Winter) Let $\mathcal{D}$ be a unital separable C^*-algebra. We say that $\mathcal{D}$ is strongly self-absorbing (s.s.a) if $\mathcal{D} \neq \mathbb{C}$ and there is an isomorphism $\varphi \colon \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ such that

$$\varphi \simeq_{a.u} id \otimes 1_{\mathcal{D}}.$$

Strongly self-absorbing C^*-dynamical systems

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Recently, Szabo introduced s.s.a C^*-dynamical systems.
In particular, he generalized some results of Tom-Winter, Kirchberg and Dadarlat-Winter on s.s.a algebras to the equivariant context.

DEFINITION. (Szabo) Let $\gamma\colon G \curvearrowright \mathcal{D}$ be an action of a locally compact group G on a separable unital C^*-algebra $\mathcal{D}$. We say that γ is an s.s.a action if the equivariant factor first embedding

$$id \otimes 1_{\mathcal{D}}\colon (\mathcal{D}, \gamma) \to (\mathcal{D} \otimes \mathcal{D}, \gamma \otimes \gamma)$$

is approximately G-unitarily equivalent to an isomorphism.

$C(X)$-algebras

DEFINITION. Let A be a unital C^*-algebra and X be a compact Hausdorff space. A is a $C(X)$-algebra, if there exists a unital $*$-homomorphism

$$\mu\colon C(X) \to \mathcal{Z}(A)$$

from $C(X)$ onto the center of the multiplier of A.

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Theorem (a classical result) Let A be a C^*-algebra and X be a locally compact Hausdorff space. If there is a continuous map $\sigma\colon Prim(A) \to X$, then A is a $C_0(X)$-algebra.

$C_0(X)$-algebras arise naturally!

Theorem (a classical result) Let A be a C^*-algebra and X be a locally compact Hausdorff space. If there is a continuous map $\sigma\colon Prim(A) \to X$, then A is a $C_0(X)$-algebra.

- The converse of above theorem holds!

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- Let $Y \subset X$ be a closed subset, then

$$J_Y := C_0(X \setminus Y).A$$

  is a closed two sided ideal,

$$A(Y) = A_Y := A/J_Y.$$

  Note that A_Y is a $C(Y)$-algebra and a $C(X)$-algebra in the obvious way.

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and we say that A_x is the fiber of A on x.

It is interesting to study which properties are shared between all fibers and the $C(X)$-algebra!

THEOREM (Dadarlat-Winter, 2008)

Let A be a separable unital $C(X)$-algebra over a finite dimensional compact metrizable space X. Suppose that all fibers are isomorphic to the same strongly self-absorbing C^*-algebra $\mathcal{D}$. Then A and $C(X) \otimes \mathcal{D}$ are isomorphic as $C(X)$-algebras.

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Theorem (Hirshberg-Winter-White, 2007)

Let A be a separable $C(X)$-algebra over a finite dimensional compact metrizable space X and $\mathcal{D}$ be a strongly self-absorbing C^*-algebra. Then A is $\mathcal{D}$-stable if and only if all fibers are $\mathcal{D}$-stable.

Equivariant Choi-Effros Theorem provides us a powerful tool to obtain equivariant generalization of previous results.

F-Gardella-Thomsen

Let G be a second countable, locally compact group and $\gamma\colon G \curvearrowright \mathcal{D}$ be a unitarily regular strongly self-absorbing action on a separable, unital C^*-algebra $\mathcal{D}$. Let (A, α) be a separable G-$C(X)$-algebra where X is a compact space of finite covering dimension. Then all fibers of A are $(\mathcal{D}, \gamma)$-stable if and only if (A, α) is $(\mathcal{D}, \gamma)$-stable.

F-Gardella-Thomsen

Let G be a second countable, locally compact Hausdorff group and X be a compact metrizable space of finite covering dimension. Let $(\mathcal{D}, \gamma)$ be a strongly self-absorbing, unitarily regular G-C^*-algebra and (A, α) be a separable unital G-$C(X)$-algebra. Suppose that each fiber A_x is G-equivariantly isomorphic to $\mathcal{D}$, for each $x \in X$, then (A, α) and $(C(X) \otimes \mathcal{D}, \alpha \otimes \iota)$ are isomorphic as G-$C(X)$-algebras.