Algebraic and other entropies

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Introduction

Joint work with D. Dikranjan and A. Giordano Bruno

A short introduction to algebraic entropy for action of amenable groups
Particular emphasis on the Addition Theorem for ent and h_{alg}

- Algebraic entropy has several variants (ent, h_{rk}, h_{alg}, $\tilde{\mathrm{ent}}$, ...)
- Introduced (for one endomorphism) in [Adler, Kohnheim, McAndrew '65] and [Weiss '75], became popular after [Dikranjan, Goldsmith, Salce, Zanardo '09]
- Inspired by similar notions in:
  - information theory (Shannon),
  - ergodic theory (Kolmogorov and Sinai),
  - topological dynamics (Peters and Weiss)
- We will consider dynamical entropies (i.e.: entropies of endomorphisms)

Contents

Algebraic entropy of one endomorphism

- B Abelian group
- ϕ endomorphism of B
- $\ell(X) := \log|X|$

Definition (Entropy)

$$H_\ell(\phi, B_0) := \lim_{n\to\infty} \frac{\ell(\sum_{i=0}^{n-1} \phi^i(B_0))}{n}$$

$$\mathrm{ent}(\phi) := \sup\Big\{ H_\ell(\phi, B_0) : B_0 < B \text{ finite subgroup} \Big\}$$

$$h_{alg}(\phi) := \sup\Big\{ H_\ell(\phi, B_0) : 0 \in B_0 \subseteq B \text{ finite subset} \Big\}.$$

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$H_\ell(\phi, B_0)$ is the average growth of ℓ along the partial trajectory of B_0.

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$H_\ell(\phi, B_0)$ is the average growth of ℓ along the partial trajectory of B_0.
The limit in the definition of H_ℓ exists (Fekete's Lemma).

Basic properties

- ℓ is additive: if $0 \to A \to B \to C \to 0$ is an exact sequence of (Abelian) groups,
$$\ell(B) = \ell(A) + \ell(C)$$
- An Abelian group B with an endomorphism ϕ is the same object as a $\mathbb{Z}[X]$-module ($X * b = \phi(b)$)
- ent behaves as a "rank" function on $\mathbb{Z}[X]$-modules (more precisely, it is a length function for torsion $\mathbb{Z}[X]$-modules)
- $\mathrm{ent}(\phi)$ depends only on the restriction of ϕ to the torsion of B (since every finite group is torsion)
- Two isomorphic $\mathbb{Z}[X]$-modules have the same entropy (ent is an invariant)
- If ϕ is an automorphism, then $\mathrm{ent}(\phi) = \mathrm{ent}(\phi^{-1})$
- If B is torsion, then $\mathrm{ent}(\phi) = h_{alg}(\phi)$.

Amenable (semi-)groups

G cancellative semi-group.
G is right amenable if it there exists a Følner net $(F_i)_{i \in I}$ for G:
each F_i is a nonempty subset of G,

$$\forall g \in G \ \lim_{i \to \infty} \frac{|F_i g \,\Delta\, F_i|}{|F_i|} = 0.$$

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Equivalently, if it exists a nonempty *-finite set $F \subset {}^*G$ (in the non-standard universe) such that

$$\forall g \in G \quad \frac{|Fg \,\Delta\, F|}{|F|} \text{ is infinitesimal}$$

(Notice that g varies only among standard elements)

Algebraic entropy of a (semi-)group action

- B Abelian group
- G right-amenable cancellative semi-group
- α left action of G on B by group endomorphisms.

Definition (Entropy)

$$F * B_0 := \sum_{g \in F} \alpha(g)(B_0), \qquad \text{the partial trajectory}$$

$$H_\ell(\alpha, B_0) := \lim_{i \to \infty} \frac{\ell(F_i * B_0)}{|F_i|}, \qquad \text{the average growth of } \ell$$

$$\mathrm{ent}(\alpha) := \sup\left\{ H_\ell(\alpha, B_0) : B_0 < B \text{ finite subgroup} \right\}$$

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- An Abelian group B with an endomorphism ϕ is the same object as a $\mathbb{Z}[X]$-module ($X * b = \phi(b)$)
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Basic properties

- An Abelian group B with an action of a semigroup G is the same object as a $\mathbb{Z}[G]$-module ($g * b = \alpha(g)(b)$)
- ent behaves as a “rank” function on $\mathbb{Z}[G]$-modules (more precisely, it is a length function for torsion $\mathbb{Z}[G]$-modules)
- $\operatorname{ent}(\alpha)$ depends only on the action on the torsion of B
- $\operatorname{ent}(\alpha) = h_{alg}(\alpha)$ when B is torsion
- Two isomorphic $\mathbb{Z}[G]$-modules have the same entropy (ent is an invariant)
- If G is commutative group, then α^{-1} is also a left action, and $\operatorname{ent}(\alpha^{-1}) = \operatorname{ent}(\alpha)$.
- $h_{top}(G \curvearrowright \hat{A}) = \operatorname{ent}(G \curvearrowright A)$, where $\hat{A}$ is the Pontryagin dual

The Addition Theorem for algebraic entropy

Statement

- G is a right-amenable cancellative semigroup
- B is an Abelian group
- α is a (left) action of G on B (by group endomorphisms)
- A is an α-invariant subgroup of B (for every $g \in G$, $g * A \subseteq A$)
- α_A is the induced action of G on A
- $\alpha_{B/A}$ is the induced action of G on B/A.

Theorem (Addition "Theorem")

$$h_{alg}(\alpha) = h_{alg}(\alpha\restriction_A) + h_{alg}(\alpha_{B/A}).$$

Theorem (AT: equivalent formulation)

If
$$0 \to A \to B \to C \to 0$$
is an exact sequence of $\mathbb{Z}[G]$*-modules, then*

$$h_{alg}(B) = h_{alg}(A) + h_{alg}(C).$$

Idea of proof

We verified the proof of the Addition Theorem when

- either B is torsion,
- or under the following simplifying assumption on G (which include the cases when $G = \mathbb{Z}$ or $G = \mathbb{N}$):

Monotiling condition

- G is countable;
- There exists a Følner sequence $(F_n)_{n \in \mathbb{N}}$ for G such that $1 \in F_n$ and each F_n tiles F_{n+1}:
  that is, F_{n+1} is the disjoint union of translates of F_n.

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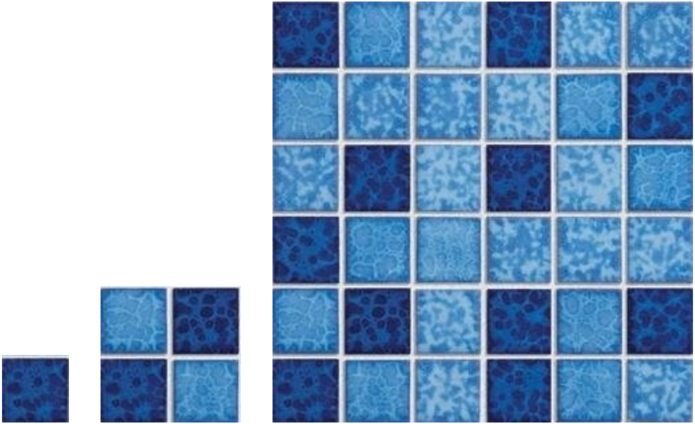

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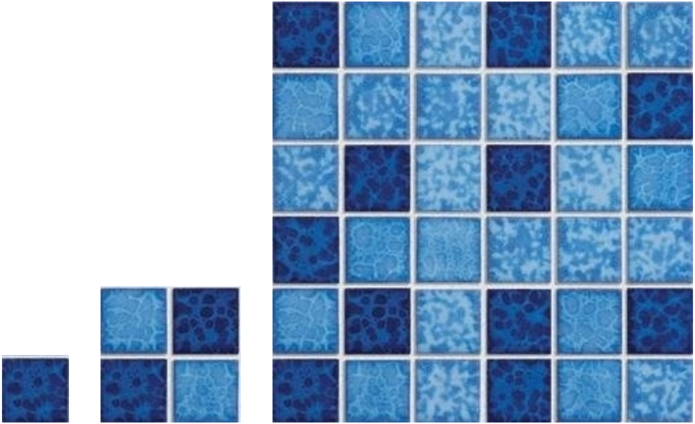

Exercise

$\mathbb{Q}$ satisfies the monotiling condition.

The (relative) size

B Abelian group, $0 \in X, Y, \dots \subseteq B$

$$\begin{aligned} \mu(X \mid Y) &:= \text{number of translated of } Y \text{ needed to cover } X \\ \ell(X \mid Y) &:= \log \mu(X \mid Y) \\ \ell(X) &:= \ell(X \mid 0) = \log|X| \end{aligned}$$

Basic properties of the function ℓ

- $\ell \geq 0$
- $\ell(X \mid Y)$ is increasing in X and decreasing in Y
- $\ell(X \mid X) = 0$
- $\ell(X + X' \mid Y + Y') \leq \ell(X \mid Y) + \ell(X' \mid Y')$
- $\ell(X \mid Y) \leq \ell(X \mid Z) + \ell(Z \mid Y)$.

Localization of ℓ

Given $A < B$, define

$$\ell_A(X \mid Y) := \ell(X \mid Y + A)$$
$$\ell_A(X) := \ell(X \mid A) = \log|X/A|.$$

Notice that $\ell_0 = \ell$.

Given $A < B$, and $0 \in Z \subseteq A$, we have

- $\ell(X + Z) \geq \ell_A(X) + \ell(Z)$

Moreover:

- if X is finite, then there exists $0 \in Z \subseteq A$ finite such that
$$\ell_A(X) = \ell(X \mid Z);$$
- if $\ell_A(X)$ is finite, then there exists $0 \in X' \subseteq X$ finite such that
$$\ell_A(X \mid X') = 0 \quad \text{(that is, } X \subseteq A + X'\text{)}.$$

Proof of AT

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Let $0 \in X \subset B$ be finite and $\bar{X} := \pi(X) \subseteq B/A$.

Fix $\varepsilon > 0$. W.l.o.g., we may assume that, for every $n \in \mathbb{N}$,

$$H(\alpha; X) \simeq_\varepsilon \frac{\ell(F_n * X)}{|F_n|}$$

$$H(\alpha_{B/A}; \bar{X}) \simeq_\varepsilon \frac{\ell(F_n * \bar{X})}{|F_n|} = \frac{\ell_A(F_n * X)}{|F_n|}$$

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Let $0 \in Z \subset A$ finite, such that

$$\ell_A(F_0 * X) = \ell(F_0 * X \mid Z).$$

Let $N \in \mathbb{N}$ such that

$$\frac{\ell(F_N * Z)}{|F_N|} \simeq_\varepsilon H(\alpha; Z).$$

By using the monotiling condition and the properties of ℓ:

$$\frac{\ell(F_N * X \mid F_N * Z)}{|F_N|} \leq \frac{\ell(F_0 * X \mid F_0 * Z)}{|F_0|} \leq \frac{\ell(F_0 * X \mid Z)}{|F_0|} = \frac{\ell_A(F_0 * X)}{|F_0|}$$

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Therefore,

$$H(\alpha; X) \simeq_\varepsilon \frac{\ell(F_N * X)}{|F_N|} \leq \frac{\ell(F_N * X \mid F_N * Z)}{|F_N|} + \frac{\ell(F_N * Z)}{|F_N|} \leq$$
$$\leq \frac{\ell_A(F_0 * X)}{|F_0|} + \frac{\ell(F_N * Z)}{|F_N|} \leq h_{alg}(\alpha_{B/A}) + h_{alg}(\alpha\restriction_A) + 2\varepsilon.$$

Since the above is true for every X and every ε, the conclusion follows. □

Simpler version of ℓ

The proof could be simplified if we could use, in place of ℓ:

$$\ell'(X \mid Y) := \log\Big(\frac{|X+Y|}{|Y|}\Big)$$

We can use ℓ' if:

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We can use ℓ' if: Either the following holds:

Conjecture

Let B be an Abelian group. Let X, Y, Z be finite symmetric subsets of B containing 0. Then,

$$\frac{|X+Y|}{|Y|} \geq \frac{|X+Y+Z|}{|Y+Z|}$$

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Or when B is torsion: then, all the sets X, Y, Z appearing in the proof can be assumed to be subgroups, and

$$\ell(X \mid Y) = \log|(X+Y)/Y| = \ell'(X \mid Y)$$

Rank entropy

Definition

R integral domain, with field of fraction R_0
rk_R rank function on R-modules: $\mathrm{rk}_R(B) = \dim_{R_0}(B \otimes_R R_0)$,

$$\mathrm{rk}_R(B \mid A) := \mathrm{rk}_R((A+B)/A).$$

G amenable (cancellative semi-)group with Følner net $(F_i)_{i\in I}$;
α action of G on R-module B.

Entropy: $H_{\mathrm{rk}_R}(\alpha, B_0) := \lim_{i\to\infty} \frac{\mathrm{rk}_R(F_i * B_0)}{|F_i|}$

$$h_{\mathrm{rk}_R}(\alpha) := \sup\left\{ H_{\mathrm{rk}_R}(\alpha, B_0) : B_0 < B, \mathrm{rk}_R(B) < \infty \right\}$$

h_{rk_R} satisfies AT.

Example

Let $G = \mathbb{N}$ and B be an $R[X]$-module. Then,

$$h_{\mathrm{rk}_R}(B) = \mathrm{rk}_{R[X]}(B)$$

Dynamical entropy of matroids

(X, rk) finitary matroid/pregeometry
$(\mathrm{rk} : \mathcal{P}(X) \to \mathbb{N} \cup \{\infty\}$ is the rank function)
α action of a group G on (X, r)

Example

X field, $\mathrm{rk}(\bar{a}) := \mathrm{tr.deg.}(\bar{a})$, G groups of field automorphisms of X.

Definition (Entropy)

$$H_{\mathrm{rk}}(\alpha, \bar{b}) := \lim_{i \to \infty} \frac{\mathrm{rk}(\bigcup_{g \in F_i} g * \bar{b})}{|F_i|}$$

$$h_{\mathrm{rk}}(\alpha) := \sup\left\{ H_{\mathrm{rk}}(\alpha, \bar{b}) : \bar{b} \subseteq X, \bar{b} \text{ finite} \right\}$$

Lemma

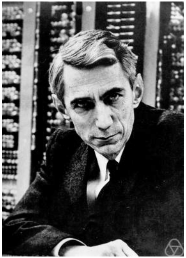
If $G = \mathbb{Z}^m$, then h_{rk} is a matroid on X.

Example

K field, σ field automorphism.
"Closure operator for h_{rk}" = "differential-algebraic closure"
$x \in cl^{\sigma}(A)$ iff there exists a differential-algebraic polynomial
$f(X) := p(X, X^{\sigma}, X^{\sigma^2}, \dots)$
with coefficients in the field generated by $A \cup \sigma(A) \cup \sigma^2(A) \cup \dots$
such that $f(x) = 0$.

When $K \models ACFA$,

$$U(\bar{a}) = h_{\mathrm{rk}}(\bar{a}) \cdot \omega + \mathrm{o}(\omega)$$



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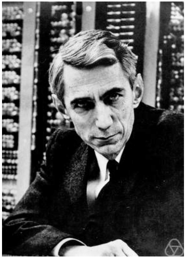
> *"You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, nobody knows what entropy really is, so in a debate you will always have the advantage."*

"



Claude Shannon