BANACH SPACE ACTIONS, ADMISSIBLE ALGEBRAS AND GROUP UNIFORMITIES.

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Joint ongoing work with
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- **SEMIGROUP COMPACTIFICATIONS.**
- **REPRESENTABILITY.**
- **UNIFORMITIES.**

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Definition. Given a (Hausdorff) topological group G a *semigroup compactification* (X, ψ) of G is defined as a pair, where X is a semigroup with a compact Hausdorff topology and $\psi : G \to X$ is a continuous homomorphism with dense image such that:

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• There is a correspondence between certain subalgebras of $C_b(\mathcal{G})$ and semigroup compactifications.

• Given a norm–closed subalgebra $\mathcal{A}$ of $\mathrm{C}_b(\mathcal{G})$ containing the constant functions, a linear functional $\mu \in \mathcal{A}^*$ is a *mean* on $\mathcal{A}$ if $\mu(f) \in [\inf_{\mathcal{G}} f, \sup_{\mathcal{G}} f]$, for every $f \in \mathcal{A}$. A mean $\mu \in \mathcal{A}^*$ is *multiplicative* if $\mu(f \cdot g) = \mu(f)\mu(g)$ for all $f, g \in \mathcal{A}$. We denote the set of all multiplicative means on $\mathcal{A}$ by $\mathbb{MM}(\mathcal{A})$.

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• A norm–closed subalgebra $\mathcal{A}$ of $\mathrm{C}_b(\mathcal{G})$ containing the constant function is called *m–admissible* if it is left–translations invariant, and it is *left*–m–*introverted*, i.e. for every $\nu \in \mathrm{MM}(\mathcal{A})$ and every $f \in \mathcal{A}$ the function $\mathrm{T}_\nu f$, defined below belongs to $\mathcal{A}$:

$$(\mathrm{T}_\nu f)(x) := \langle \nu, L_x f \rangle \qquad (x \in \mathcal{G}),$$

• Given an admissible algebra $\mathcal{A}$ we can define a product on $\mathbb{M}(\mathcal{A})$ by $\mu\nu(f) := \mu(\mathrm{T}_\nu f)$. It can be shown that $\mathbb{M}(\mathcal{A})$ with this product and its w^*–topology is a semigroup compactification of $\mathcal{G}$ which we shall denote by $\mathcal{G}^{\mathcal{A}}$. Moreover, the functions in $\mathcal{A}$ are precisely the functions which can be continuously extended to $\mathcal{G}^{\mathcal{A}}$.

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• The *coefficients* of the representation ψ are the complex–valued functions defined on $\mathcal{G}$ by the formula:

$$g \longmapsto \langle \psi(g)x, y^* \rangle,$$

where $x \in B$, $y^* \in B^*$ and $\langle \cdot, \cdot \rangle$ denotes the dual pairing of B and B^*.

• The WOT of $\mathrm{Is}(B)$ is defined as the weakest topology which makes the functions

$$T \longmapsto \langle T(x), y^* \rangle$$

continuous for all $x \in B$ and $y^* \in B^*$.

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- c_0 is not reflexively representable (Jorge Galindo and SF).
- The *Tsirelson* space T is not reflexively representable (Alexander Berenstein, Itaï Ben Iaacov and SF).

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• Notice that similar results relating representability to certain admissible algebra exist for many classes of Banach spaces, such as Hilbert spaces, Asplund spaces, Rosenthal spaces (see the work of Eli Glasner and Michael Megrelishvili who also also considered general dynamical systems).

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Theorem *Let* $\mathcal{G}$ *be a topological group and suppose that the coefficients of representations of* $\mathcal{G}$ *as isometries of Banach spaces which belong to some* nice *nice* $\mathcal{F}$*, generate the topology of* $\mathcal{G}$*, then* $\mathcal{G}$ *embeds into* $\mathrm{Is}(B)$ *for some* $B \in \mathcal{F}$*.*

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Remark. Many admissible algebras can be defined in this way.

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• It is a simple observation that a pre–pseudometric is a pseudometric if and only if it satisfies the triangle inequality.
• A pre–pseudometric h is a pseudometric precisely when its *triangle deficiency* $\Theta_h : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$, defined by

$$\Theta_h(a,b) = \sup\{h(x,z) : h(x,y) \leq a,\ h(y,z) \leq b,\ x,y,z \in X\},$$

satisfies $\Theta_h(a,b) \leq a+b$, for every $a,b \in \mathbf{R}^+$. We shall be interested in pre–pseudometrics whose triangle deficiency function enjoys the following continuity property.

• A symmetric increasing function $g : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is a *TD–function* if the following condition is satisfied:

$$\text{Whenever } y < t, \text{there exists } \delta > 0 \text{ such that } g(\delta, y + \delta) < t.$$

Here we call a function $g : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *increasing* if $x \leq x'$ and $y \leq y'$ imply $g(x, y) \leq g(x', y')$.

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• In order for the triangle deficiency function Θ_h to satisfy the above condition it suffices that the pre–pseudometric h satisfies a condition related to the notion of uniform equivalence which is stated in the following definition.

Definition. Two pre–pseudometrics h_1 and h_2 on the set X are *uniformly equivalent* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h_j(x, y) < \delta$ implies $|h_i(x, z) - h_i(z, y)| < \varepsilon$, for $i, j \in \{1, 2\}$ and for every $x, y, z \in X$. A pre–pseudometric h defined on X is *locally continuous* if it is uniformly equivalent to itself, namely if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $h(x, y) < \delta$ implies $|h(x, z) - h(z, y)| < \varepsilon$, for every $x, y, z \in X$.

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• The triangle deficiency of every locally continuous pre–pseudometric is a TD–function.

Definition. A continuous increasing function $f : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ is a *correction function* for $g : \mathbf{R}^+ \times \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ provided

$$(f \circ g)(a, b) \leq f(a) + f(b),$$

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• Every TD–function admits a correction function.

• A pre–pseudometric h is locally continuous if and only if it is uniformly equivalent to a pseudometric. This pseudometric can be taken of the form $f \circ h$, where f is a correction function for Θ_h.

Theorem. *Let $\mathcal{G}$ be a topological group. An m–admissible algebra $\mathcal{A} \subset \mathrm{UC}(\mathcal{G})$ determines the topology of $\mathcal{G}$ if and only if the uniformity of $\mathcal{G}$ is induced by a family of $\mathcal{A}$–pseudometrics, i.e. pseudometrics $\{\delta_\iota : \iota \in I\}$ with the property that, for every $\iota \in I$ the corresponding function "distance from the identity" $\delta_e(\cdot) := \delta(e, \cdot)$ belongs to $\mathcal{A}$. If the topology of $\mathcal{G}$ is induced by a left–invariant metric d the algebra $\mathcal{A}$ determines the topology of $\mathcal{G}$ if and only if there exists a left–invariant $\mathcal{A}$–metric uniformly equivalent to d.*

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Proof. (Idea for the metric case.) Construct the pre–metric

$$h(x, y) := \sum_{n=0}^{\infty} \frac{f_n\left(x^{-1}y\right)}{2^{n+1}}$$

using functions $f_n \in \mathcal{A}$ which generate the topology of $\mathcal{G}$ and correct it. $\square$

Thank you!