Inertial methods in group theory

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intrinsic algebraic entropy

Recall For a subset F of an abelian group $(A, +)$, $\varphi \in End(A)$ and $n \in \mathbb{N}$, consider the series of n-**th trajectories** of F under the action of φ:

$$T_n = T_n(\varphi, F) := F + \varphi(F) + \ldots + \varphi^{n-1}(F).$$

Each factor T_{n+1}/T_n is an epic image of $F + \varphi(F)/F$.
Therefore, when F is finite, the sequence $|T_{n+1}/T_n|$ stabilizes and there exists

$$\mathcal{H}_{alg}(\varphi, F) := \lim_{n\to\infty} \frac{1}{n} \log |T_n(\varphi, F)|$$

The **entropy** of φ is defined as

$$ent(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : F \textbf{ finite } \textbf{\textit{subgroup}} \text{ of } A\}..$$

We do not need F to be finite but just that the quotient $F + \varphi(F)/F$ is finite.

Dikranian, Giordano Bruno, Salce and Virili (2015) defined the *intrinsic* algebraic entropy (which works in non-periodic groups as well)

$$\widetilde{ent}(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : |F + \varphi(F)/F| < \infty\}.$$

the **inertial** correspondence as a dynamical property

Let φ denote an endomorphism and H a subgroup of any group G.

Definition

- H is called *φ-inert* if $\varphi(H) \cap H$ has finite index in $\varphi(H)$.
  that is $\varphi(H)$ is *almost contained* in H,
- An endomorphism φ of a group G is called an *inertial endomorphism* if H is φ-inert for each subgroup H of G.
- A subgroup H of an (abelian) group G is called *fully inert subgroup* if H is φ-inert for each endomorphism φ of G.

• all subgroups of $\mathbb{Q}$ are fully inert.
• if H is an open compact subgroup of a locally compact totally disconnected group G, then H *is φ-inert for each continous endomorphisms* φ of G and one considers the scale of φ

$$s_G(\varphi) = \min_H |H^\varphi : (H \cap H^\varphi)|.$$

where H ranges over all open compact subgroups of G

fully inert algebraic entropy

The **entropy** of φ is defined by

$$\mathcal{H}_{alg}(\varphi, F) := \lim_{n\to\infty} \frac{1}{n} \log |T_n(\varphi, F)|$$

$$ent(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : F \textbf{ finite } \textbf{\textit{subgroup}} \text{ of } A\}..$$

the *intrinsic* algebraic entropy works in non-periodic groups as well and is

$$\widetilde{ent}(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, F) : H \text{ is } \varphi\text{-inert}\}.$$

• Castellano and Giordano Bruno (2017) studied the case in the category of vector spaces.

• Goldsmith - Salce (2015) defined the *fully inert algebraic entropy*

$$\tilde{ent}(\varphi) := \sup\{\mathcal{H}_{alg}(\varphi, H) : H \text{ is fully inert}\}.$$

and showed that $\widetilde{ent}(\varphi)$ can be computed by using $\tilde{ent}(\bar{\varphi})$ of the induced endomorphism $\bar{\varphi}$ of φ-invariant sections of G.

Can we define *intrinsic* entropy even in the non-abelian or non-(locally finite) case???.
If H is a φ-inert subgroup of a group G,
then the index $|H^\varphi : (H \cap H^\varphi)|$ is just the minimal size of a subset X of H such that $H^\varphi = (H \cap H^\varphi)X^\varphi$ (setwise product), hence $H^\varphi \subseteq HX^\varphi$.
By using the trajectories of finite transverals like $Y := X^\varphi$ we have .

Let φ be an endomorphism of a group G. If H is a φ-inert subgroup of G with $t := |H^\varphi : (H \cap H^\varphi)|$, then for every $n \in \mathbb{N}$, the setwise product $T_n := H \cdot H^\varphi \dots H^{\varphi^n}$ is contained in a setwise product HY_n, where $Y_n \subseteq G$ is finite.

If t_n :=the smallest possible size of $Y_n \subseteq G$ with $T_n \subseteq HY_n$, then $t_n \leq t^n$ and

$$\exists \widetilde{\mathcal{H}}(\varphi, H) := \limsup \frac{\log t_n}{n}$$

fully invariant vs fully inert

Definition Two subgroups H and K are called *commensurable* if their intersection $H \cap K$ has finite index in both of them

Fact A subgroup which is commensurable with a φ-inert subgroup is φ-inert.

- Subgroups which are *commensurable with a fully invariant subgroups* are fully inert.

Theorem (Dikranjan, Salce and Zanardo, 2014)

A fully inert subgroup H of a **free abelian** *group A is commensurable with a fully invariant subgroup of A (that is with nA for some $n \in \mathbb{N}$).*

fully invariant subgroups of p-groups have a complicated structure, however

Theorem (Goldsmith, Salce and Zanardo, 2014)

- if A is a **direct sum of cyclic p-groups** *A, then any fully inert subgroup of A is commensurable with a fully invariant subgroup of A.*

- there exist p-groups A that contain fully inert subgroups which are **not** *commensurable with any fully invariant subgroup of A.*

• **Fact+** A subgroup which is commensurable with a φ-invariant subgroup is uniformly fully φ-inert, that is $\exists n$ s.t. $|(H+\varphi(H))/H| \le n \ \forall \varphi \in End(G)$.

Theorem (Bergman, Lenstra, 1989)

For a subgroup H of a (non-commutative) group G t.f.a.e.
- H is commensurable with some normal subgroup of G
- H is uniformly inert w.r.t. inner automorphisms, that is
$\exists n \in \mathbb{N}$ *s.t.* $\forall g \in G \ \ |H^g : (H^g \cap H)| \le n$ *(where* $H^g = g^{-1}Hg$*).*

• The subgroup $H := \langle (1,2,3)\ ,\ (4,5,6)\ ,(7,8,9)\ \ldots \rangle$ of $\mathrm{Alt}(\mathbb{N})$ is inert, but not commensurable to any normal subgroup of $\mathrm{Alt}(\mathbb{N})$ (which is simple).

Corollary

For a subgroup H of an abelian group A the following are equivalent:
- there is n s.t. $|\gamma(H)/\gamma(H) \cap H| \le n$ *for each* automorphism γ *of A;*
- H is commensurable with a characteristic subgroup of A.

CONJECTURE

a subgroup H of an abelian group A is uniformly fully inert if and only if H is commensurable with a fully invariant subgroup of A.

Problem: *Characterize the abelian groups in which each uniformly fully inert subgroup is commensurable to some fully invariant subgroup.*

Theorem (Dardano, Dikranjan, Rinauro 2018)

For a divisible abelian group D the following are equivalent:

- *every fully inert subgroup of D is commensurable with a fully invariant subgroup of D;*
- *either $r_0(D) = 0$ or $r_0(D)$ is infinite;*
- *every fully inert subgroup of D is uniformly inert.*

Corollary

• *every uniformly fully inert subgroup of a* **divisible** *abelian group D is commensurable with a fully invariant subgroup of D.*

non-commutative groups in which all subgroups are inert

A subgroup H of a group G is *inert* if the index $|H : H^g \cap H|$ is finite for all $g \in G$. Clearly normal and finite subgroups are inert.

Consider the following properties for a group G:

(TIN) $\forall H \leq G\ \forall g \in G\ \exists n = n_{g,H} \in \mathbb{N}\ \ |H : (H \cap H^g)| \leq n$;
(each subgroup is inert, G is inertial)

(CN) $\forall H \leq G\ \exists n = n_H \in \mathbb{N}\ \forall g \in G\ \ |H : (H \cap H^g)| \leq n$
(each subgroup is uniformly inert w.r.t. inner automorphisms)

(BCN) $\exists n \in \mathbb{N}\ \forall H \leq G\ \forall g \in G\ \ |H : (H \cap H^g)| \leq n$;

(BIN) $\forall g \in G\ \exists n = n_g \in \mathbb{N}\ \forall H \leq G\ \ |H : (H \cap H^g)| \leq n$. *still OPEN*

Note that Tarski groups have even (CN).

Theorem (Casolo, D, Rinauro, *J. Algebra* 496, 2018)

- (PILF) Let G a group whose periodic quotients are locally finite.
If G is a CN-group (i.e. each subgroup is uniformly inert),
then G is finite-by-abelian-by-finite.
- For each prime p there is a nilpotent p-group with the property BCN, which is neither abelian-by-finite nor finite-by-abelian.

Fact (generalized) soluble CN-groups may be described via abelian-by-finite CF-groups, that is groups G in which subgroup H is core-finite, that is H/H_G is finite.

Theorem *(CDR 2018)* Let G be a finite-by-abelian-by-finite group.
i) G is CN if and only if it is finite-by-CF.
ii) G is BCN if and only if it is finite-by-BCF.
iii) if G is periodic, then G is BCN if it is CN.
iv) A locally graded BCN-group is finite-by-abelian-by-finite.

Definitions:
- *abelian-by-finite* = there is an abelian (characteristic) subgroup with finite index.
- *locally finite* = every finitely generated subgroup is finite indeed.
- *locally graded* = every non-trivial finitely generated subgroup has a non-trivial finite quotient. This is a rather large class of groups, containing all residually finite groups and all locallyâĂŞ(soluble by finite) groups.
- PILF contains *generalized radical* groups, i.e. groups with an ascending normal series whose factors are either locally nilpotent or finite.

the RING of inertial endomorphisms of an abelian group

Inertial endomorphisms of any abelian group A form a ring, say $\mathcal{I}\mathrm{End}(A)$, containing the ideal $F(A)$ of endomorphisms with finite image and the subring $M(A)$ of the multiplications. .

Theorem (D, Rinauro, 2014)

The ring $\mathcal{I}\mathrm{End}(A)$ of inertial endomorphisms of any abelian group A is commutative modulo its ideal $F(A)$ *formed by the endomorphisms of A with finite image.*

Say that an abelian p-group is *critical* when the max divisible subgroup $Div(A) \neq 0$ has finite rank and $A/Div(A)$ is bounded but infinite.

Theorem (D, Rinauro, 2014)

Let A be an abelian p-group.

1. *If A is non-critical then $\mathcal{I}\mathrm{End}(A) = F(A) + M(A)$.*
2. *If A is critical, then $\mathcal{I}\mathrm{End}(A) = M(A) + F(A) + N$, where $N \simeq \mathbb{Z}(p^e)$ and p^e is the exponent of $A/D(A)$.*

If $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p^\infty$, then $\varphi = (-id \oplus id) \in \mathcal{I}\mathrm{End}(A) \setminus (F(A) + M(A))$.

the GROUP of bi-inertial automorphisms of any group

Definition

If γ is an automorphism of a group G we say that:
• γ is *bi-inertial* if H and H^γ are commensurable for each $H \leq G$
(i.e. both γ and γ^{-1} are inertial) ;
• *bi-inertial automorphisms of any group G form a subgroup $\mathcal{I}\mathrm{Aut}(G)$ of $\mathrm{Aut}(G)$, which may contain any Tarski group or any centreless FC-group.*

• *groups of inertial automorphisms can have a complicated structure, but if the underlying group is abelian the picture is not so terrible.*

Theorem (D, Rinauro, 2016)

Let $\mathcal{I}\mathrm{Aut}(A)$ be the group generated by all inertial automorphisms of an abelian group A.

1. *If A is torsion-free, then $\mathcal{I}\mathrm{Aut}(A)$ is abelian.*
2. *if A is periodic , $\mathcal{I}\mathrm{Aut}(A)$ is centre-by-(locally finite).*
3. *$\mathcal{I}\mathrm{Aut}(A)$ is metabelian-by-(locally finite).*
4. *$\mathcal{I}\mathrm{Aut}(A)$ is NOT (locally nilpotent)-by-(locally finite) when $A = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}$.*

THE END

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References: see the survey paper by D.Dikranjan, UD, S.Rinauro
Inertial Properties in Groups,
International Journal of Group Theory (2018), 7, no. 3, pp. 17-62.

Definition

An endomorphism γ of a group G is called *finitary* if the subgroup $C_G(\gamma) := \{x \in G \mid x^\gamma = x\}$ has finite index in G.

- a group is an FC-group if and only if all its inner automorphisms are finitary.
- in any case, finitary automorphism form a group, say $\mathrm{FAut}(G)$.

Theorem

Let $\mathrm{FAut}(G)$ be the group of all finitary automorphisms of any group G. Then:

1. $\mathrm{FAut}(G)$ *is (locally finite), when $G = A$ is abelian* (Wehrfritz 2002).
2. $\mathrm{FAut}(G)$ *is abelian-by-(locally finite)* (Belyaev and Shved, 2009-11);
3. $\mathrm{FAut}(G)$ *is locally (centre-by-finite) hence (locally finite)-by-abelian* (Zalesskii 1975);
4. $\mathrm{FAut}(G)$ *is locally finite iff* $\mathrm{FAut}(G) \cap Inn(G)$ *is locally finite* (Menegazzo and Robinson 1987)
5. *there is a normal series* $1 \leq \Gamma_1 \leq \Gamma_2 \leq \mathrm{FAut}(G)$ *such that* (Shved 2017):
   1. Γ_1 *is nilpotent of class al most 4; ;*
   2. Γ_2/Γ_1 *is periodic and has an ascending central series of type at most ω;*
   3. $\mathrm{FAut}(G)/\Gamma_2$ *is embeddable into a direct product of finitary linear groups over prime fields.*