On topological entropy on non compact spaces

Jose S. Cánovas[1]

[1] Universidad Politécnica de Cartagena

DAGT, Udine, July 2018

Outline

Compact case

- Let X be a compact and let $f : X \to X$ be continuous.
- The topological entropy of f is a non-negative number $h(f)$ which often is taken as a measure of the dynamical complexity of f.
- Adler, Konheim and McAndrew's definition:
  - $\mathcal{A} = \{A_i\}$ a finite open cover of X.
  - $\mathcal{N}(\mathcal{A})$ the minimum number of elements from $\mathcal{A}$ to cover X.
  - $\vee_{i=0}^{n-1} f^{-i}(\mathcal{A}) = \{\cap_{i=0}^{n-1} f^{-i}(A_{j_i}) : A_{j_i} \in \mathcal{A}\}$, $f^{-i} = (f^i)^{-1}$, and $f^i = f \circ f^{i-1}$, $i \geq 1$, f^0 is the identity on X.
  - $h(f, \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(\vee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right)$.
- $h_{AKM}(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is a finite open cover of } X\}$.

Compact case: metric

- Let (X, d) metric and let $f : X \to X$ be uniformly continuous.
- Bowen's definition:
    - Let K be a compact subset of X and fix $\varepsilon > 0$ and $n \in \mathbb{N}$.
    - A subset $S \subset K$ is said to be (n, ε, K, f)–separated if for any $x, y \in S$, $x \neq y$, there is $k \in \{0, 1, ..., n-1\}$ such that $d(f^k(x), f^k(y)) > \varepsilon$.
    - Denote by $s_n(\varepsilon, K, f)$ the cardinality of a maximal (n, ε, K, f)–separated subset S.
    - The topological entropy of f on K is

$$h_d(f, K) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(\varepsilon, K, f),$$

- The topological entropy of f is

$$h_d(f) = \sup\{h_d(f, K) : K \subset X \text{ compact}\}.$$

Compact case: metric

- Let (X, d) compact and metric and let $f : X \to X$ be continuous. Then

$$h_{AKM}(f) = h_d(f).$$

- In particular, topological entropy does not depend on the metric generating the topology on X.
- Variational principle

$$h_{AKM}(f) = sup\{h_\mu(f) : \mu \in \mathcal{M}(X, f)\} = h_{VP}(f),$$

where $\mathcal{M}(X, f)$ is the set of Borel invariant measures of f $(\mu(A) = \mu(f^{-1}(A))$ for all Borel set $A)$ and $h_\mu(f)$ is the metric (Kolmogorov-Sinai) entropy of f.

Compact case: Basic properties

(TE1) Let $f, p : X \to X$ and $g : Y \to Y$ be continuous maps on the topological spaces X and Y, respectively and let $\varphi : X \to Y$ be continuous and such that $\varphi \circ f = g \circ \varphi$.

1. If φ is surjective, then $h_{AKM}(f) \geq h_{AKM}(g)$.
2. If φ is injective, then $h_{AKM}(f) \leq h_{AKM}(g)$.
3. If φ is bijective, in particular if it is an homeomorphism, then $h_{AKM}(f) = h_{AKM}(g)$.

(TE2) Power formula. For $n \in \mathbb{N}$, $h_{AKM}(f^n) = nh_{AKM}(f)$. Additionally, if f is an homeomorphism then $h_{AKM}(f) = h_{AKM}(f^{-1})$, and therefore $h_{AKM}(f^n) = nh_{AKM}(f)$ for all $n \in \mathbb{Z}$.

(TE3) Product formula. $h_{AKM}(f \times g) = h_{AKM}(f) + h_{AKM}(g)$.

(TE4) Commutativity formula. $h_{AKM}(f \circ p) = h_{AKM}(p \circ f)$.

(TE5) If $Y \subset X$ is compact and invariant by f, then $h_{AKM}(f|_Y) \leq h_{AKM}(f)$. In addition, if $X = \cup_{i=1}^{k} X_i$ and $f(X_i) \subseteq X_i$, then $h_{AKM}(f) = \max\{h_{AKM}(f|_{X_i}) : i = 1, 2..., k\}$

Framework

- Now, let $f : X \to X$ be a continuous map on a non necessarily compact (metric) space.
- Fixed $x \in X$, we denote by $\mathrm{Orb}_f(x)$ the forward orbit of x under f, that is, the sequence $f^n(x)$ for $n \in \mathbb{N}$.
- The set of accumulation points of $\mathrm{Orb}_f(x)$ is the ω-limit set $\omega_f(x)$.
- This ω–limit set is closed and strictly invariant by f, and then we can define the following invariant subsets of X:

$$K_f = \{x \in X : \omega_f(x) \text{ is compact and non empty}\},$$

$$C_f = \{x \in X : \omega_f(x) \text{ is non empty and non compact}\},$$

$$E_f = \{x \in X : \omega_f(x) \text{ is empty}\}.$$

Two strategies/Two scenarios/Particular cases

- Defining by compactifications.
- Defining by compact subsets.

- Non metric.
- Metric.

- Locally compact spaces.
- Shift spaces.
- One dimensional maps.
- ...

Hofer's approach

- Hofer first definition:

$$h_H(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ finite open cover of } X\}$$

- Second one: X is a completely regular space and $f^* : X^* \to X^*$, the unique continuous extension of f to the Stone–Cech compactification X^* of X. Then

$$h_H^*(f) = h_{AKM}(f^*) \leq h_H(f).$$

- Both definitions are equivalent when X is normal, but not true in general (Fedeli's counterexample).

Hofer's approach/Compactifications approach

- In what follows X is normal. Then (Hasselblatt, Nitecki and Propp): Let $f : X \to X$ be a continuous map such that $E_f \neq \emptyset$. Then $h_H(f) = +\infty$.
- Problems with properties (T1)–(T5).
- Hofer example. We consider the map $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(n) = n + 1$ for all $n \in \mathbb{Z}$ holds that $h_H^*(f) = h_H(f) = \infty$.
- We can define

$$h^*(f) = \inf\{h_{AKM}(f^*) : f^* \text{ is a extension of } f\}.$$

Compact atoms approach/Inspired by Misiurewicz topological conditional entropy

- Denote by $\mathcal{K}(X)$ the family of compact subsets of X and let $\mathcal{A}$ be an open cover of X.
- For $K \in \mathcal{K}(X)$, we define the number

$$\mathcal{N}\left(K, \vee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right)$$

as the smallest cardinality of a subcover of $\vee_{i=0}^{n-1} f^{-i}(\mathcal{A})$ such that the union of its elements contains K. Then

$$h(f, K, \mathcal{A}) = \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{N}\left(K, \vee_{i=0}^{n-1} f^{-i}(\mathcal{A})\right)$$

and

$$h(f, K) = \sup\{h(f, K, \mathcal{A}) : \mathcal{A} \text{ open cover of } X\}.$$

Finally

$$h_{MC}(f) = \sup\{h(f, K) : K \in \mathcal{K}(X)\}.$$

Compact atoms approach

- Let $\mathcal{K}(X,f)$ be the family of compact subsets K of X which are invariant by f, that is, $f(K) \subseteq K$.
- The restricted map $f|_K$ is the properly defined in a compact space.
- Then (Cánovas and Rodriguez)

$$h_{CR}(f) = \sup\{h_{AKM}(f|_K) : K \in \mathcal{K}(X,f)\}.$$

- Good properties (T1)–(T5), except for (TE1)(1) and (TE5) maximum formula.
- In general

$$h_{CR}(f) \leq h^*(f) \leq h_H(f).$$

$$h_{CR}(f) \leq h_{MC}(f) \leq h_H(f).$$

Friedland's approach

- Let X^* be a compactification of X. Then $(X^*)^{\mathbb{N}}$ is compact and the shift map $\sigma : (X^*)^{\mathbb{N}} \to (X^*)^{\mathbb{N}}$, given by $\sigma(x_n) = (x_{n+1})$, is continuous. Consider

$$X^f = \{(x_i)_{i=0}^{\infty} : x_i = f(x_{i-1}) \text{ with } i = 1, 2, ...\} \subset (X^*)^{\mathbb{N}}.$$

Then $\sigma(X^f) \subset X^f$ and Friedland topological entropy is

$$h_F(f) = h(\sigma|_{\mathrm{Cl}(X^f)}).$$

- Then

$$h_F^*(f) = \inf\{h_F(f) : X^* \text{ is a compactification of } X\}.$$

When X is compact $h_{AKM}(f) = h_{AKM}(\sigma|_{X^f})$.

Proper maps (Patrao)

- An open cover $\mathcal{A} = \{A_i : i \in \mathcal{I}\}$ of a Haussdorf topological space X is co–compact if for any $A_i \in \mathcal{A}$ the subset $X \setminus A_i$ is compact. Co–compact covers have finite subcovers.
- A continuous map $f : X \to X$ is said to be proper if $f^{-1}(K)$ is compact for all K compact subset of X. Then $f^{-1}(\mathcal{A})$ is a co–compact cover of X.
- Define

$$h_P(f) = \sup\{h(f, \mathcal{A}) : \mathcal{A} \text{ is a co} - \text{compact open cover of } X\}.$$

- For proper maps $h_{CR}(f) \leq h_P(f) \leq h_H(f)$.

Few comments

- Everything is more rich due to metrics. Bowen's definitions, invariant measures...
- Definitions are metric dependent.
- In general, no Lebesgue number for covers, which implies problems to relate Bowen and open covers definitions.

Countable shift maps

- Let $\Sigma = \{0, 1, ...\}$ be a countable alphabet (vertices).
- Consider a matrix $\mathbf{A} = (a_{ij})$ with infinite entries of 0's and 1's and let $\mathcal{S} = \{(x_n) \in \Sigma^{\mathbb{Z}} : a_{x_n x_{n+1}} = 1, n \in \mathbb{Z}\}$ and let $\sigma : \mathcal{S} \to \mathcal{S}$ be the shift map given by $\sigma(x_n) = (x_{n+1})$.
- We assume that $\mathcal{S}$ is connected, that is, for any $i, j \in \Sigma$, there are $n \in \mathbb{N}$ and a block $[i, x_1, ..., x_{n-1}, j]$ contained in an element of $\mathcal{S}$.
- If the alphabet Σ is finite, then $\mathcal{S}$ is compact and σ [or $(\mathcal{S}, \sigma)$], is a subshift of finite type. If Σ is infinite, then $\mathcal{S}$ is not compact and the Gurevic entropy of σ is then defined by

$$h_G(\sigma) = \sup\{h_{AKM}(\sigma|_K) : \sigma|_K \text{ is a subshift of finite type}\}.$$

Countable shift maps

- Σ is locally compact, we can compactify it by a point $\Sigma^* = \Sigma \cup \{\infty\}$ and we can take $\mathcal{S}^*$ the closure of $\mathcal{S}$ in $(\Sigma^*)^{\mathbb{Z}}$ and the continuous extension $\sigma^* : \mathcal{S}^* \to \mathcal{S}^*$. The nice result of Gurevic states that

$$h_G(\sigma) = h_{AKM}(\sigma^*).$$

- Since the subshifts of finite type of $\mathcal{S}$ are in the family of $\mathcal{K}(\mathcal{S}, \sigma)$, then we obtain that

$$h_G(\sigma) = h_{CR}(\sigma) = h_{MC}(\sigma) = h_{AKM}(\sigma^*) = h^*(\sigma) = h_{VP}(\sigma).$$

- In addition, one can introduce the totally bounded metric d_G on $\mathcal{S}$. Then $h_G(\sigma) = h_{d_G}(\sigma)$. In addition

$$h_G(\sigma) = h_D(\sigma) = \inf\{h_d(\sigma) : d \text{ metric}\}.$$

Continuous real maps

- Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = 2x$ (d is standard Euclidean distance).
- The dynamics of ϕ is quite simple: $x_0 = 0$ is the unique fixed point of f and then, if $x > 0$, the sequence $\phi^n(x)$ converges monotically to $+\infty$ and if $x < 0$, then $\phi^n(x)$ goes to $-\infty$.
- On the other hand, $h_d(\phi) = \log 2$.
- We have that $h_{CR}(\phi) = h_{AKM}(\phi^*) = h^*(\phi) = 0$.

Continuous real maps

- $f:\mathbb{R}\to\mathbb{R}$ continuous.
- Let $[a,b]\subset\mathbb{R}$ and consider the natural retraction $\tau_{[a,b]}:\mathbb{R}\to[a,b]$ by

$$\tau_{[a,b]}(x)=\begin{cases} a, & \text{if } x<a,\\ x, & \text{if } x\in[a,b],\\ b, & \text{if } x>b.\end{cases}$$

Define $f_{[a,b]}:[a,b]\to[a,b]$ by $f_{[a,b]}=\tau_{[a,b]}\circ f|_{[a,b]}$, where $f|_{[a,b]}:[a,b]\to\mathbb{R}$ is the restriction of f to $[a,b]$. Hence,

$$h_{CR}(f)=\sup\{h_{AKM}(f_{[a,b]}):a,b\in\mathbb{R},a<b\}.$$

- $h_{CR}(f)>0$ if and only if f^k has a horsehoe for some k. Relationship with chaos.

Continuous real maps

- $h_{CR}(f) > 0$ if and only if f^k has a horsehoe for some k. Relationship with chaos.
- If f is piecewise monotone (for all compact subinterval $[a, b]$, there is a finite number of subintervals of $[a, b]$ such that f is monotone on such intervals), then $h_{CR}(f) \leq h_G(\sigma)$ for a suitable countable shift.
- If f has Markov property, there is a countable shift such that $h_{CR}(f) = h_G(\sigma)$.

Some references

- R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy,* Trans. Amer. Math. Soc. **114** (1965), 309–319.
- R. Bowen, *Entropy for group endomorphism and homogeneous spaces,* Trans. Amer. Math. Soc. **153** (1971), 401–414.
- J. S. Cánovas and J. M. Rodríguez, *Topological entropy of maps on the real line,* Topology and its Applications **153** (2005), 735-746.
- S. Friedland, *Entropy of polynomial and rational maps*, Annals of Mathematics, **133** (1991), 359–368.
- B. M. Gurevic, *Topological entropy of a countable Markov chain,* (Russian) Dokl. Akad. Nauk SSSR **187** (1969), 715–718.English translation [Soviet Math. Dokl. **10** (1969), 911–915].
- M. Handel and B. Kitchens, *Metrics and entropy for non–compact spaces,* Israel Journal of Mathematics **91** (1995), 253–271. (Appendix by D. J. Rudolph)
- B. Hasselblatt, Z. Nitecki and J. Propp, *Topological entropy for nonuniformly continuous maps,* Discrete and Continuous Dynamical Systems **22** (2008), 201–213.
- J. E. Hofer, *Topological entropy for noncompact spaces*, Michigan Math. J. **21** (1974), 235–242.
- M .Misiurewicz, *Topological conditional entropy,* Studia Mathematica **55** (1976), 175–200.
- M. Patrao, *Entropy and its variational principle for non-compact metric spaces*, Ergod. Th. & Dynam. Sys. **30** (2010), 1529–1542.
- D. Dikranjan, M. Sanchis and S. Virili, *New and old facts about entropy in uniform spaces and topological groups,* Topology and its Applications **159** (2012), 1916-1942.