Complete semitopological semilattices

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Dynamical methods in Algebra, Geometry and Topology, July 4, 2018

Definition

A semilattice if a commutative semigroup of idempotents,

for example, the interval $[0, 1]$ with operation of min or max.

A topologized semilattice is a semilattice endowed with a topology.

Definition

Let $\mathcal{C}$ be a class of topological semilattices. A topologized semilattice X is defined to be

- *e:$\mathcal{C}$-closed* if for any isomorphic topological embedding $f : X \to Y \in \mathcal{C}$ the image $f(X)$ is closed in Y;
- *h:$\mathcal{C}$-closed* if for any continuous homomorphism $f : X \to Y \in \mathcal{C}$ the image $f(X)$ is closed in Y;
- *p:$\mathcal{C}$-closed* if each closed subsemilattice $Z \subseteq X$ is h:$\mathcal{C}$-closed.

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In the role of the class $\mathcal{C}$ we consider the classes:

- TS of Hausdorff topological semilattices;
- sTS of Hausdorff semitopological semilattices.

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- topological if the semilattice operation $\cdot : X \times X \to X$ is continuous;
- semitopological if the semilattice operation $\cdot : X \times X \to X$ is separately continuous.

Definition

A topologized semilattice X is called

- chain-finite if each chain in X is finite;
- chain-compact if each closed chain in X is compact;
- k-complete if each non-empty chain C of X has $\inf C \in \overline{C}$ and $\sup C \in \overline{C}$.

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Previous results

Theorem (Stepp, 1975).

Each chain-finite semilattice X is h:TS-closed.

Question (Stepp, 1975).

Is there exists a e:TS-closed topological semilattice which is not h:TS-closed?

Theorem (Gutik, Repovs, 2008).

For a linear topological semilattice X the following conditions are equivalent:

- X is e:TS-closed;
- X is h:TS-closed;
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Answer to the question of Stepp (B., Gutik, 2012).

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Main results

First, we improve the Stepp's result:

Theorem (Banakh, B. 2017)

For any homomorphism $h : X \to Y$ from a chain-finite semilattice X to a semitopological semilattice Y satisfying the separation axiom T_1, the image $h(X)$ is closed in Y.
Consequently, each chain-finite semilattice is p:sTS-closed.

Theorem (Banakh, B. 2017)

For a Hausdorff semitopological semilattice X the following conditions are equivalent:

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Open Problem

Is it true that each k-complete semitopological semilattice is h:sTS-closed?

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Partial answers on the problem

Theorem (Banakh, B., 2018)

For any continuous homomorphism $h : X \to Y$ from a k-complete topologized semilattice X to a sequential Hausdorff semitopological semilattice Y the image $f(X)$ is closed in Y.

Theorem (Banakh, B., Ravsky, 2018)

For any continuous homomorphism $h : X \to Y$ from a k-complete topologized semilattice X to a functionally Hausdorff semitopological semilattice Y the image $f(X)$ is closed in Y.

Theorem (Banakh, B., Ravsky, 2018)

For each functionally Hausdorff semitopological semilattice X the natural partial order $\{(x, y) \in X \times X \mid xy = x\}$ is a closed subset of $X \times X$.

Open Problem

Is there exists a Hausdorff semitopological semilattice with a non-closed natural partial order?

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Banakh topology on topologized semilattices

Definition

Let (X, τ) be a topologized semilattice.

- The small semilattice topology τ_L on X is generated by the base which consists of open subsemilattices of (X, τ);
- the Banakh topology τ_B on X is generated by the subbase consisting of complements to closed subsemilattices of (X, τ).

Remark

If (X, τ) is a semitopological semilattice then the topologized semillatices (X, τ_B) and (X, τ_L) are semitopological as well.

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Definition

A topologized semilattice X is called

- a *U-semilattice* if for each $x \in X$ and for each open neighborhood U of x there exists a point $y \in U$ such that $x \in \mathrm{Int}_X {\uparrow} y$;
- a *V-semilattice* if for any points $x \not\leq y \in X$ there exists a point $z \in X \setminus {\downarrow} y$ such that $x \in \mathrm{Int}_X {\uparrow} z$.

Theorem

For a compact Hausdorff semitopological semilattice (X, τ) the following statements are equivalent:

- $\tau = \tau_L$;
- $\tau = \tau_B$;
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Thank You for attention!