Closedness versus completeness in topological groups

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Motivation: complete = absolutely closed

- A metric space X is complete if and only if X is closed in any metric space containing X as a subspace.
- A uniform space X is complete if and only if X is closed in any uniform space containing X as a subspace.
- A topological group X is Raikov complete iff X is closed in any topological group containing X as a topological subgroup.
- A Tychonoff space X is compact if and only if X is closed in each Tychonoff topological space containing X as a subspace.

The completeness is a property of the uniform structure whereas the absolute closedness has a categorical nature and has sense in situations where no natural uniform structure exists.

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f:$\mathcal{C}$-closed topologized semigroups

We shall work in categories whose objects are topologized semigroups and morphisms are continuous homomorphisms of topologized semigroups.

A *topologized semigroup* is a semigroup endowed with a topology. Let $\mathcal{C}$ be a class of topologized semigroups and f be a class of homomophisms between topologized semigroups.

Principal Definition

A topologized semigroup X is defined to be *f:$\mathcal{C}$-closed* if for any homomorphism $f : X \to Y \in \mathcal{C}$ in the class f the image $f(X)$ is closed in Y.

Principal Problem

Characterize f:*$\mathcal{C}$-closed topologized semigroups.*

In this talk I will concentrate on f:$\mathcal{C}$-closed topological groups. Serhii Bardyla will talk about f:$\mathcal{C}$-closed topologized semilattices.

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Concrete examples of classes f

We shall be interested in four instances of f:$\mathcal{C}$-completeness.

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- *p:$\mathcal{C}$-closed* if each closed subsemigroup $Z \subset X$ is h:$\mathcal{C}$-closed, which means that for any continuous homomorphism $h : Z \to Y \in \mathcal{C}$ the image $h(Z)$ is closed in Y.

For any topologized semigroup we have the implications:

$$\text{p:}\mathcal{C}\text{-closed} \Rightarrow \text{h:}\mathcal{C}\text{-closed} \Rightarrow \text{i:}\mathcal{C}\text{-closed} \Rightarrow \text{e:}\mathcal{C}\text{-closed}$$

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Concrete examples of classes $\mathcal{C}$ of topologized groups

In the role of the classes $\mathcal{C}$ we shall consider the classes:

- TG of Hausdorff topological groups;
- rTG of Hausdorff right-topological groups;
- pTG of Hausdorff paratopological groups;
- sTG of Hausdorff semitopological groups;
- qTG of Hausdorff quasitopological groups.

A topologized group X is called

- paratopological if its binary operation $X \times X \to X$, $(x, y) \mapsto xy$, is continuous;
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Besides the classes of topologized groups, we shall also consider the classes:

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It is clear that TS $\subset$ pTS $\subset$ sTS.

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Raikov and Bardyla-Gutik-Ravsky Theorems

A topological group is called *complete* if it is complete in its two-sided uniformity. Such topological group are also called Raikov-complete because of the following characterization.

Theorem (Raikov, 1946)

A topological group X is complete if and only if X is closed in each topological group containing X as a topological subgroup.

Theorem (Bardyla, Gutik, Ravsky, 2017)

A topological group X is complete if and only if X is closed in each quasitopological group containing X as a topologized subgroup.

All topologized semigroups mentioned in this talk are assumed to be Hausdorff.

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For a topological group X the following conditions are are equivalent:

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Characterizing compact topological groups

Theorem (Gutik, 2014; Banakh, 2017)

For a topological group X the following conditions are equivalent:

1. *X is compact;*
2. *X is* e:sTS*-closed;*
3. *X is* e:rTG*-closed.*

Problem

Is each e:sTG*-closed topological group compact?*

Theorem (B., 2017)

Each e:sTG*-closed topological group has compact center.*

Characterizing compact topological groups

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3. *X is* e:rTG*-closed.*

Problem

Is each e:sTG*-closed topological group compact?*

Theorem (B., 2017)

Each e:sTG*-closed topological group has compact center.*

Characterizing compact topological groups

Theorem (Gutik, 2014; Banakh, 2017)

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f:$\mathcal{C}$-closedness properties of topological groups

complete	$\Longleftarrow$									compact
$\Updownarrow$										$\Updownarrow$
e:qTG-closed	$\Leftrightarrow$	e:TG-closed	$\Leftarrow$	e:pTG-closed	$\Leftarrow$	e:TS-closed	$\Leftarrow$	e:pTS-closed	$\Leftarrow$	e:sTS-closed
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Updownarrow$
i:qTG-closed	$\Rightarrow$	i:TG-closed	$\Leftarrow$	i:pTG-closed	$\Leftarrow$	i:TS-closed	$\Leftarrow$	i:pTS-closed	$\Leftarrow$	i:sTS-closed
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Updownarrow$
h:qTG-closed	$\Rightarrow$	h:TG-closed	$\Leftarrow$	h:pTG-closed	$\Leftarrow$	h:TS-closed	$\Leftarrow$	h:pTS-closed	$\Leftarrow$	h:sTS-closed
$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Uparrow$		$\Updownarrow$
p:qTG-closed	$\Rightarrow$	p:TG-closed	$\Leftarrow$	p:pTG-closed	$\Leftarrow$	p:TS-closed	$\Leftarrow$	p:pTS-closed	$\Leftarrow$	p:sTS-closed

Minimal versus i:TG-closed topological groups

A topological group X is *minimal* if every bijective continuous homomorphism $h : X \to Y$ to a top. group is a top. isomorphism.

This definition implies that

each minimal complete topological group is i:TG-*closed*.

Problem

Is each i:TG-*closed topological group minimal?*

The answer is affirmative in two partial cases:

Theorem (B., 2017)

An Abelian topological group X is compact iff X is i:TG-*closed (i.e., X is complete in each weaker Hausdorff group topology).*

Theorem

An ω-narrow topological group X of countable pseudocharacter is i:TG-*closed if and only if X is complete and minimal.*

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Definition

A topological group X has *compact exponent* if for some $n \in \mathbb{N}$ the set $nX = \{x^n : x \in X\}$ has compact closure in X.

Theorem (B., Ravsky)

For an Abelian topological group X the following conditions are equivalent:

1. X *is* e:pTG*-closed;*
2. X *is* e:TS*-closed;*
3. X *is* e:pTS*-closed;*
4. X *is complete and has compact exponent.*
5. X *is complete and for any injective continuous homomorphism* $f : X \to Y$ *to a topological group* Y *the quotient group* $\overline{f(X)}/f(X)$ *is periodic.*

Topological groups of compact exponent

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Selected characterizations of compact topological groups

Theorem (B., 2017)

An Abelian topological group is compact iff X is i:TG*-closed.*

Theorem (Dikranjan, Uspenski, 1998)

A nilpotent topological group X is compact iff X is h:TG*-closed.*

Theorem (Dikranjan, Uspenski, 1998)

A solvable topological group X is compact iff X is p:*TG-closed iff each closed subgroup of X is* h:TG*-closed.*

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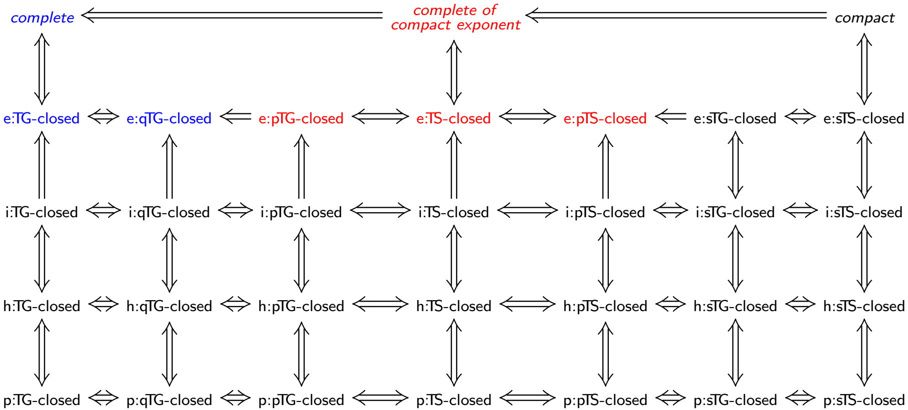
Closedness properties of Abelian topological groups



Much more information on this subject can be found in the papers:

- T.Banakh, *Categorically closed topological groups*, Axioms **6**:3 (2017) 23.
- S. Bardyla, O. Gutik, A. Ravsky, *H-closed quasitopological groups*, Topology Appl. **217** (2017), 51–58.
- O.V. Gutik, *On closures in semitopological inverse semigroups with continuous inversion*, Algebra Discrete Math. **18**:1 (2014), 59–85.
- A. Ravsky, *On H-closed paratopological groups*, Visnyk Lviv Univ., Ser. Mekh.-Mat. **61**, (2003), 172–179.

References (Lviv)

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- D. Dikranjan, M. Megrelishvili, *Minimality conditions in topological groups*, Recent Progress in General Topology III, Hart, K.P., van Mill, Jan, Simon, P (Eds.) Springer Verlag (Atlantis Press), Berlin, (2014), 229–327.
- D. Dikranjan, V.V. Uspenskij, *Categorically compact topological groups*, J. Pure Appl. Algebra, **126** (1998), 149–168.
- A. Lukàcs, *Compact-like topological groups*, Heldermann Verlag, 2009.

Thank you!